Faculty of Science, Technology, Engineering and Mathematics
MS327 Deterministic and stochastic dynamics

MS327 diagnostic quiz

Are you ready for MS327?

This diagnostic quiz is designed to help you decide if you are ready to study MS327. This document also contains some advice on preparatory work that you may find useful before starting MS327 (see below and page 20). The better prepared you are for MS327 the more time you will have to enjoy the mathematics, and the greater your chance of success.

The topics which are included in this quiz are those that we expect you to be familiar with before you start the module. If you have previously studied MST210 or MST224 then you should be familiar with most of the topics covered in the quiz.

We suggest that you try this quiz first without looking at any books, and only look at a book when you are stuck. You will not necessarily remember everything; for instance, for the calculus questions you may need to look up a set of rules for differentiation or integration or use the table of standard derivatives or integrals provided on page 2. This is perfectly all right, as such tables are provided in the Handbook for MS327. You need to check that you are able to use them though.

If you can complete the quiz with only the occasional need to look at other material, then you should be well prepared for MS327. If you find some topics that you remember having met before but need to do some more work on, then you should consider the suggestions for refreshing your knowledge on page 20.

Try the questions now, and then see the notes on page 20 of this document to see if you are ready for MS327. (The answers to the questions begin on page 9.)

Do contact your Student Support Team via StudentHome if you have any queries about MS327, or your readiness to study it.

Tables of standard derivatives and integrals

Function	Derivative
a	0
x^a	ax^{a-1}
e^{ax}	ae^{ax}
$\ln(ax)$	$\dfrac{1}{x}$
$\sin(ax)$	$a\cos(ax)$
$\cos(ax)$	$-a\sin(ax)$
$\tan(ax)$	$a\sec^2(ax)$
$\cot(ax)$	$-a\,\mathrm{cosec}^2(ax)$
$\sec(ax)$	$a\sec(ax)\tan(ax)$
$\mathrm{cosec}(ax)$	$-a\,\mathrm{cosec}(ax)\cot(ax)$
$\arcsin(ax)$	$\dfrac{a}{\sqrt{1-a^2x^2}}$
$\arccos(ax)$	$-\dfrac{a}{\sqrt{1-a^2x^2}}$
$\arctan(ax)$	$\dfrac{a}{1+a^2x^2}$
$\mathrm{arccot}(ax)$	$-\dfrac{a}{1+a^2x^2}$
$\mathrm{arcsec}(ax)$	$\dfrac{a}{\lvert ax\rvert\sqrt{a^2x^2-1}}$
$\mathrm{arccosec}(ax)$	$-\dfrac{a}{\lvert ax\rvert\sqrt{a^2x^2-1}}$

Function	Integral
a	ax
$x^a\ \ (a \neq -1)$	$\dfrac{x^{a+1}}{a+1}$
$\dfrac{1}{ax+b}$	$\dfrac{1}{a}\ln\lvert ax+b\rvert$
e^{ax}	$\dfrac{1}{a}e^{ax}$
$\ln(ax)$	$x(\ln(ax)-1)$
$\sin(ax)$	$-\dfrac{1}{a}\cos(ax)$
$\cos(ax)$	$\dfrac{1}{a}\sin(ax)$
$\tan(ax)$	$-\dfrac{1}{a}\ln\lvert\cos(ax)\rvert$
$\cot(ax)$	$\dfrac{1}{a}\ln\lvert\sin(ax)\rvert$
$\sec(ax)$	$\dfrac{1}{a}\ln\lvert\sec(ax)+\tan(ax)\rvert$
$\mathrm{cosec}(ax)$	$\dfrac{1}{a}\ln\lvert\mathrm{cosec}(ax)-\cot(ax)\rvert$
$\sec^2(ax)$	$\dfrac{1}{a}\tan(ax)$
$\mathrm{cosec}^2(ax)$	$-\dfrac{1}{a}\cot(ax)$
$\dfrac{1}{x^2+a^2}$	$\dfrac{1}{a}\arctan\left(\dfrac{x}{a}\right)$
$\dfrac{1}{(x-a)(x-b)}$	$\dfrac{1}{a-b}\ln\left\lvert\dfrac{a-x}{x-b}\right\rvert$
$\dfrac{1}{\sqrt{x^2+a^2}}$	$\ln(x+\sqrt{x^2+a^2}\,)\quad or\quad \mathrm{arcsinh}\left(\dfrac{x}{a}\right)$
$\dfrac{1}{\sqrt{x^2-a^2}}$	$\ln\lvert x+\sqrt{x^2-a^2}\,\rvert\quad or\quad \mathrm{arccosh}\left(\dfrac{x}{a}\right)$
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\arcsin\left(\dfrac{x}{a}\right)$

Diagnostic quiz questions

Differentiation

Question 1

Differentiate the following functions with respect to x.

(a) $f(x) = x^5 + \sin x + e^x + 4$ (b) $f(x) = \cos 2x + \ln(4x)$

(c) $f(x) = (x^2 + 4)\sin 5x$ (d) $f(x) = \dfrac{x^2 + 2x - 1}{3x + 4}$

(e) $f(x) = \sin(3x^2 + 2)$

Question 2

Find the second derivative with respect to x of $f(x) = 3x^4 + 2\sin 3x$.

Question 3

Find and classify the stationary points of $y = x^3 + 3x^2 - 24x + 10$.

Integration

Question 4

Calculate the following integrals.

(a) $\displaystyle\int (3x^5 + 6)\,dx$ (b) $\displaystyle\int_1^2 \left(x^2 + \frac{2}{x}\right)dx$ (c) $\displaystyle\int x^2(x^3 + 5)^{10}\,dx$

(d) $\displaystyle\int \sin^5 2x \cos 2x\,dx$ (e) $\displaystyle\int x\cos 3x\,dx$

Ordinary differential equations

Question 5

Find the general solution of the differential equation

$$\frac{dy}{dx} = x^3 + 3.$$

Question 6

Find the particular solution of the differential equation

$$\frac{dy}{dx} + \sin 2\pi x = 0,$$

that satisfies the condition $y(0) = \dfrac{1}{\pi}$.

Question 7

Find the particular solution of the differential equation

$$\frac{dy}{dx} = \frac{xy}{x^2+1},$$

where $y > 0$, which satisfies $y(1) = 5\sqrt{2}$.

Question 8

Find the general solution of the differential equation

$$\frac{dy}{dx} + \frac{2y}{x} = x.$$

Question 9

Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = e^{2x}.$$

Question 10

Find the particular solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 5x + 8$$

that satisfies $y(0) = 4$ and $\frac{dy}{dx}(0) = 9$.

Partial differentiation

Question 11

Find the first partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ of the function $f(x,y) = 2x^3y^4$.

Question 12

Find the second-order partial derivatives of the function

$$u(x,y) = \cos(4x+3y),$$

and confirm that $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$.

Question 13

Use the Chain Rule to find $\frac{du}{dt}$, where $u(x,y) = \ln\left(x^2y^4\right)$ and $x = \cos 2t,\ y = \sin t$.

Matrices

Question 14

Let $\mathbf{A} = \begin{pmatrix} 5 & 4 \\ 2 & 0 \\ 1 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 1 & 5 \\ 2 & 0 \end{pmatrix}$.

Calculate each of the following, or state why the calculation is not possible.

(a) $\mathbf{AB}$ (b) $\mathbf{BA}$ (c) $2\mathbf{A} + 3\mathbf{B}$ (d) $2\mathbf{B} + \mathbf{C}$

Question 15

Calculate the determinants of the following matrices.

(a) $\begin{pmatrix} 5 & 2 \\ 3 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & 1 & 0 \\ 3 & 4 & 2 \\ 1 & 5 & 1 \end{pmatrix}$

Question 16

Find the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 1 & 8 \\ -1 & -5 \end{pmatrix}$.

Question 17

Find the eigenvalue of the matrix $\begin{pmatrix} 26 & 1 & -5 \\ 4 & 20 & -4 \\ 4 & 8 & 8 \end{pmatrix}$ that corresponds to the eigenvector $\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$.

Vector algebra and calculus

Question 18

If $\mathbf{a} = 3\,\mathbf{i} + 2\,\mathbf{j} - 5\,\mathbf{k}$ and $\mathbf{b} = 2\,\mathbf{i} - \mathbf{j} - \mathbf{k}$, find

(a) $\mathbf{a} + \mathbf{b}$, $|\mathbf{a} + \mathbf{b}|$, $\mathbf{a} \cdot \mathbf{b}$, and $\mathbf{a} \times \mathbf{b}$,

(b) the angle between $\mathbf{a}$ and $\mathbf{b}$, to the nearest degree.

Question 19

If $\phi(x, y, z) = x^2y + 2xz$, find $\boldsymbol{\nabla}\phi$ (also known as $\mathbf{grad}\,\phi$) at the point $(2, -2, 3)$.

Question 20

If $\mathbf{F} = x^2y\,\mathbf{i} - 2xz\,\mathbf{j} + 2yz\,\mathbf{k}$, find $\boldsymbol{\nabla} \cdot \mathbf{F}$ (also known as $\operatorname{div} \mathbf{F}$) and $\boldsymbol{\nabla} \times \mathbf{F}$ (also known as $\mathbf{curl}\,\mathbf{F}$) at the point $(1, 2, -1)$.

Multiple integration

Question 21

Evaluate the surface integral $\int_S xy\,dA$, where S is the region of the plane $z = 0$ bounded by the curve $y = x^2$ and by the line $y = x$.

Question 22

Evaluate the volume integral $\displaystyle\int_B \left(x^2y + 2xz^2\right)\,dV$, where B is the interior of the cube with faces in the planes $x = 0$, $y = 0$, $z = 0$, $x = 1$, $y = 1$ and $z = 1$.

Mechanics

Question 23

If $\mathbf{r} = 2\cos(3t)\,\mathbf{i} + 2\sin(3t)\,\mathbf{j} + (2t-1)\,\mathbf{k}$ is the position vector at time t of a moving particle, find

(a) the velocity of the particle, $\dot{\mathbf{r}}$;

(b) the speed of the particle, $|\dot{\mathbf{r}}|$;

(c) the acceleration of the particle, $\ddot{\mathbf{r}}$;

(d) a unit vector in the direction of the tangent to the trajectory of the particle at time t.

Question 24

The acceleration of particle of mass 5 kg is given by
$\mathbf{a} = 3\sin 6t\,\mathbf{i} + 3\cos 6t\,\mathbf{j} + 4\,\mathbf{k}$.

At $t = 0$ the velocity, $\mathbf{v}(0) = \mathbf{i}$ and the position, $\mathbf{r}(0) = \mathbf{j} + \mathbf{k}$.

(a) Find the magnitude of the force acting on the particle.

(b) Find the position of the particle at any time t.

Question 25

A particle connected to a spring moves in a straight line. The position of the particle along the line, x, is given by

$$\frac{d^2x}{dt^2} + 4x = 0.$$

Initially, $x(0) = 3$ and $x'(0) = 0$.

(a) What is the speed of the particle as it passes the point $x = 0$ for the first time?

(b) What is the period of the motion?

Complex numbers

Question 26

If $w = 2 + i$ and $z = 3 - 4i$ find, in Cartesian form

(a) $w + 2z$ (b) wz (c) w/z

Fourier series

Question 27

Find a Fourier series for the function $f(x) = 1 + x$ for $-1 < x < 1$.

Solutions to questions

Solution to Question 1

(a) $f'(x) = 5x^4 + \cos x + e^x$

(b) $f'(x) = -2\sin 2x + 1/x$

(c) $f'(x) = 2x\sin 5x + 5(x^2+4)\cos 5x$

(d) $f'(x) = \dfrac{(3x+4)(2x+2) - 3(x^2+2x-1)}{(3x+4)^2} = \dfrac{3x^2+8x+11}{(3x+4)^2}$

(e) $f'(x) = 6x\cos(3x^2+2)$

Solution to Question 2

$f'(x) = 12x^3 + 6\cos 3x, \quad \text{so, } f''(x) = 36x^2 - 18\sin 3x.$

Solution to Question 3

The stationary points occur when $\dfrac{dy}{dx} = 0$, that is, when $3x^2 + 6x - 24 = 0$.

This can be written $3(x-2)(x+4) = 0$, so the stationary points are $x = 2$ and $x = -4$.

At $x = 2$, $y = -18$ and at $x = -4$, $y = 90$.

The second derivative, $\dfrac{d^2y}{dx^2} = 6x + 6$.

At $x = 2$, $\dfrac{d^2y}{dx^2} = 18 > 0$, so $(2, -18)$ is a minimum.

At $x = -4$, $\dfrac{d^2y}{dx^2} = -18 < 0$, so $(-4, 90)$ is a maximum.

Solution to Question 4

(a) $\displaystyle\int (3x^5 + 6)\,dx = \frac{1}{2}x^6 + 6x + c$, where c is an arbitrary constant.

(b) $\displaystyle\int_1^2 \left(x^2 + \frac{2}{x}\right)\,dx = \left[\frac{1}{3}x^3 + 2\ln x\right]_1^2 = \frac{8}{3} + 2\ln 2 - \frac{1}{3} = \frac{7}{3} + 2\ln 2.$

(c) $\displaystyle\int x^2(x^3+5)^{10}\,dx = \frac{1}{33}(x^3+5)^{11} + c$, where c is an arbitrary constant.

(Using the substitution $u = x^3 + 5$.)

(d) $\displaystyle\int \sin^5 2x\cos 2x\,dx = \frac{1}{12}\sin^6 2x + c$, where c is an arbitrary constant.

(Using the substitution $u = \sin 2x$.)

(e) Integrating by parts gives

$$\begin{aligned}\int x\cos 3x\,dx &= x\frac{1}{3}\sin 3x - \int \frac{1}{3}\sin 3x\,dx \\ &= \frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x + c\end{aligned}$$

where c is an arbitrary constant.

Solution to Question 5

Integrating both sides with respect to x gives

$$y = \int \left(x^3 + 3\right)\, dx = \frac{1}{4}x^4 + 3x + c,$$

where c is an arbitrary constant.

Solution to Question 6

The equation can be written $\dfrac{dy}{dx} = -\sin 2\pi x$.

Integrating both sides with respect to x gives

$$y = -\int \sin 2\pi x\, dx = \frac{1}{2\pi}\cos 2\pi x + c,$$

where c is an arbitrary constant.

When $x = 0$, $y = \dfrac{1}{\pi}$, so substituting into the above gives

$$\frac{1}{\pi} = \frac{1}{2\pi} + c.$$

So, $c = \dfrac{1}{2\pi}$ and the required solution is

$$y = \frac{1}{2\pi}\left(\cos 2\pi x + 1\right).$$

Solution to Question 7

The differential equation is of first order, and the right-hand side can be written as $f(x)g(y)$, for some functions $f(x)$ and $g(y)$. So the equation can be solved using separation of variables.

Rearrange the equation so that all the terms involving y are on the left-hand side, and all those involving x are on the right:

$$\frac{1}{y}\frac{dy}{dx} = \frac{x}{x^2+1}.$$

Integrating both sides gives

$$\int \frac{1}{y}\, dy = \int \frac{x}{x^2+1}\, dx$$

hence,

$$\ln y = \frac{1}{2}\ln(x^2+1) + c,$$

where c is an arbitrary constant. So

$$y = \exp\left(\tfrac{1}{2}\ln(x^2+1) + c\right) = A\sqrt{x^2+1},$$

where $A = e^c$.

Since $y = 5\sqrt{2}$ when $x = 1$, we have $5\sqrt{2} = A\sqrt{2}$, so $A = 5$ and hence $y = 5\sqrt{x^2+1}$.

Solution to Question 8

The equation is of the form $\frac{dy}{dx} + g(x)y = f(x)$ so can be solved using the integrating factor method. The integrating factor is $p(x) = \exp\left(\int g(x)\,dx\right)$, which in this case is

$$p(x) = \exp\left(\int \frac{2}{x}\,dx\right) = \exp(2\ln x) = x^2.$$

So,

$$x^2 y = \int \left(x^2 \times x\right)\,dx = \int x^3\,dx = \frac{x^4}{4} + c,$$

where c is an arbitrary constant. Hence

$$y = \frac{x^2}{4} + \frac{c}{x^2}.$$

Solution to Question 9

The equation is linear, with constant coefficients and of second order.

The homogeneous equation is

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 0.$$

The auxiliary equation is $\lambda^2 + 2\lambda - 3 = 0$.

This can be written $(\lambda + 3)(\lambda - 1) = 0$, and solving for λ gives $\lambda = 1$ or $\lambda = -3$.

So the complementary function is $y_{\mathrm{c}} = Ae^x + Be^{-3x}$, where A and B are arbitrary constants.

The right-hand side of the inhomogeneous equation suggests a trial function is $y_{\mathrm{p}} = ae^{2x}$.

So,

$$\frac{dy_{\mathrm{p}}}{dx} = 2ae^{2x}, \quad \text{and} \quad \frac{d^2y_{\mathrm{p}}}{dx^2} = 4ae^{2x}.$$

Substituting into the differential equation, we obtain

$$4ae^{2x} + 2(2ae^{2x}) - 3ae^{2x} = e^{2x},$$

or $5ae^{2x} = e^{2x}$, so that $a = \frac{1}{5}$.

So the particular integral is $y_{\mathrm{p}} = \frac{1}{5}e^{2x}$.

The general solution is $y = Ae^x + Be^{-3x} + \frac{1}{5}e^{2x}$.

Solution to Question 10

First solve the associated homogeneous equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0.$$

The auxiliary equation is $\lambda^2 - 2\lambda + 5 = 0$,

The solutions are,

$$\lambda = \frac{2 \pm \sqrt{4 - 20}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i.$$

Hence the complementary function is

$$y_{\text{c}} = e^x(A\cos(2x) + B\sin(2x)),$$

where A and B are arbitrary constants.

Returning to the inhomogeneous equation,

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 5x + 8,$$

the form of the right-hand side suggests a trial solution of the form

$$y_{\text{p}} = ax + b.$$

So,

$$\frac{dy_{\text{p}}}{dx} = a. \quad \text{and} \quad \frac{d^2y_{\text{p}}}{dx^2} = 0.$$

Substituting into the inhomogeneous equation gives

$$0 - 2a + 5(ax + b) = 5x + 8,$$

which can be written

$$5ax + (5b - 2a) = 5x + 8.$$

Comparing coefficients gives $\quad 5a = 5 \quad$ and $\quad 5b - 2a = 8.$

So, $\quad a = 1 \quad$ and $\quad b = 2.$

The particular integral is $\quad y_{\text{p}} = x + 2.$

The general solution is $y = y_{\text{c}} + y_{\text{p}}$, that is,

$$y = e^x(A\cos(2x) + B\sin(2x)) + x + 2.$$

When $x = 0$, $y = 4$, so $4 = A + 2$, and $A = 2$.

The derivative of the general solution is (using the Product Rule)

$$\frac{dy}{dx} = e^x((A + 2B)\cos(2x) + (B - 2A)\sin(2x)) + 1.$$

When $x = 0$, $dy/dx = 9$, so

$$9 = A + 2B + 1 = 2B + 3 \quad \text{and so} \quad B = 3.$$

The particular solution required is

$$y = e^x\,(2\cos(2x) + 3\sin(2x)) + x + 2.$$

Solution to Question 11

$$\frac{\partial f}{\partial x} = 6x^2y^4, \qquad \frac{\partial f}{\partial y} = 8x^3y^3.$$

Solution to Question 12

$$\frac{\partial u}{\partial x} = -\sin(4x+3y) \times 4 = -4\sin(4x+3y)$$

$$\frac{\partial u}{\partial y} = -\sin(4x+3y) \times 3 = -3\sin(4x+3y)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(-4\sin(4x+3y)) = -4\cos(4x+3y) \times 4 = -16\cos(4x+3y)$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial}{\partial x}(-3\sin(4x+3y)) = -3\cos(4x+3y)\times 4 = -12\cos(4x+3y)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(-3\sin(4x+3y)) = -3\cos(4x+3y) \times 3 = -9\cos(4x+3y)$$

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial}{\partial y}(-4\sin(4x+3y)) = -4\cos(4x+3y)\times 3 = -12\cos(4x+3y)$$

Comparison of $\dfrac{\partial^2 u}{\partial y\,\partial x}$ with $\dfrac{\partial^2 u}{\partial x\,\partial y}$ confirms that $\dfrac{\partial^2 u}{\partial x\,\partial y} = \dfrac{\partial^2 u}{\partial y\,\partial x}$.

Solution to Question 13

The Chain Rule gives $\dfrac{du}{dt} = \dfrac{\partial u}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial u}{\partial y}\dfrac{dy}{dt}$.

We know, $\dfrac{\partial u}{\partial x} = \dfrac{1}{x^2} \times 2x = \dfrac{2}{x}$, $\quad \dfrac{dx}{dt} = -2\sin 2t$, $\quad \dfrac{\partial u}{\partial y} = \dfrac{1}{y^4} \times 4y^3 = \dfrac{4}{y}$, $\quad \dfrac{dy}{dt} = \cos t$.

So, $\dfrac{du}{dt} = \dfrac{2}{x} \times (-2\sin 2t) + \dfrac{4}{y} \times \cos t = \dfrac{4\cos t}{y} - \dfrac{4\sin 2t}{x}$.

Now substitute for x and y in terms of t:
$\dfrac{du}{dt} = \dfrac{4\cos t}{\sin t} - \dfrac{4\sin 2t}{\cos 2t} = 4\cot t - 4\tan 2t$.

Solution to Question 14

(a) $\mathbf{AB} = \begin{pmatrix} 31 & 13 \\ 6 & 2 \\ 17 & 7 \end{pmatrix}$

(b) $\mathbf{BA}$ cannot be calculated, since $\mathbf{B}$ is 2×2 and $\mathbf{A}$ is 3×2.

(c) $2\mathbf{A} + 3\mathbf{B}$ cannot be calculated, since $\mathbf{A}$ is 3×2 and $\mathbf{B}$ is 2×2.

(d) $2\mathbf{B} + \mathbf{C} = \begin{pmatrix} 7 & 7 \\ 10 & 4 \end{pmatrix}$

Solution to Question 15

(a) $\begin{vmatrix} 5 & 2 \\ 3 & 4 \end{vmatrix} = 5 \times 4 - 2 \times 3 = 14$

(b) $\begin{vmatrix} 4 & 1 & 0 \\ 3 & 4 & 2 \\ 1 & 5 & 1 \end{vmatrix} = 4\begin{vmatrix} 4 & 2 \\ 5 & 1 \end{vmatrix} - 1\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 4(4-10) - 1(3-2) = -25$

Solution to Question 16

The characteristic equation is given by

$$\begin{vmatrix} 1-\lambda & 8 \\ -1 & -5-\lambda \end{vmatrix} = 0,$$

that is $\quad (1-\lambda)(-5-\lambda) + 8 = 0 \quad$ or $\quad \lambda^2 + 4\lambda + 3 = 0.$

This can be factorised as $(\lambda+1)(\lambda+3) = 0$, hence the eigenvalues are -1 and -3.

The eigenvectors $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ associated with the eigenvalue λ are found by solving

$$\begin{pmatrix} 1-\lambda & 8 \\ -1 & -5-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0.$$

For $\lambda = -1$ this gives $\quad \begin{pmatrix} 2 & 8 \\ -1 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \quad$ which can be written as

$$\begin{aligned} 2x + 8y &= 0 \\ -x - 4y &= 0. \end{aligned}$$

These are satisfied when $x = -4y$, so an eigenvector is $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$.

For $\lambda = -3$ the equation gives $\quad \begin{pmatrix} 4 & 8 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \quad$ which can be written as

$$\begin{aligned} 4x + 8y &= 0 \\ -x - 2y &= 0. \end{aligned}$$

These are satisfied when $x = -2y$, so an eigenvector is $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

Solution to Question 17

Multiplying the matrix by the given eigenvector gives

$$\begin{pmatrix} 26 & 1 & -5 \\ 4 & 20 & -4 \\ 4 & 8 & 8 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 12 \\ 12 \\ 36 \end{pmatrix} = 12 \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}.$$

The product is 12 times the eigenvector, so the corresponding eigenvalue is 12.

Solution to Question 18

(a) $\mathbf{a}+\mathbf{b} = (3\,\mathbf{i}+2\,\mathbf{j}-5\,\mathbf{k}) + (2\,\mathbf{i}-\mathbf{j}-\mathbf{k}) = 5\,\mathbf{i}+\mathbf{j}-6\,\mathbf{k}$.

$|\mathbf{a}+\mathbf{b}| = |5\,\mathbf{i}+\mathbf{j}-6\,\mathbf{k}| = \sqrt{5^2+1^2+(-6)^2} = \sqrt{25+1+36} = \sqrt{62}$.

$\mathbf{a}\cdot\mathbf{b} = 3\times 2 + 2\times(-1) + (-5)\times(-1) = 6-2+5 = 9$.

$$\begin{aligned}\mathbf{a}\times\mathbf{b} &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & 2 & -5\\ 2 & -1 & -1\end{vmatrix}\\ &= (2\times(-1) - (-5)\times(-1))\,\mathbf{i} - (3\times(-1) - (-5)\times 2)\,\mathbf{j} +\\ &\qquad (3\times(-1) - 2\times 2)\,\mathbf{k}\\ &= -7\,\mathbf{i} - 7\,\mathbf{j} - 7\,\mathbf{k}.\end{aligned}$$

(b) The cosine of the angle θ between $\mathbf{a}$ and $\mathbf{b}$ can be calculated using $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}$.

$|\mathbf{a}| = \sqrt{3^2+2^2+(-5)^2} = \sqrt{38}$ and $|\mathbf{b}| = \sqrt{2^2+(-1)^2+(-1)^2} = \sqrt{6}$.

So, $\cos\theta = \dfrac{9}{\sqrt{38}\sqrt{6}} = \dfrac{3}{2}\sqrt{\dfrac{3}{19}}$.

Hence, $\theta = \arccos\left(\dfrac{3}{2}\sqrt{\dfrac{3}{19}}\right) = 53^\circ$ (to the nearest degree).

Solution to Question 19

The scalar field is $\phi(x,y,z) = x^2y + 2xz$.

The partial derivatives are

$$\frac{\partial\phi}{\partial x} = 2xy+2z, \qquad \frac{\partial\phi}{\partial y} = x^2, \qquad \text{and} \qquad \frac{\partial\phi}{\partial z} = 2x.$$

So, $\quad \boldsymbol{\nabla}\phi = \mathbf{grad}\,\phi = \dfrac{\partial\phi}{\partial x}\,\mathbf{i} + \dfrac{\partial\phi}{\partial y}\,\mathbf{j} + \dfrac{\partial\phi}{\partial z}\,\mathbf{k} = (2xy+2z)\,\mathbf{i} + x^2\,\mathbf{j} + 2x\,\mathbf{k}$.

At the point $(2,-2,3)$,

$\boldsymbol{\nabla}\phi = \mathbf{grad}\,\phi = (2\times 2\times(-2) + 2\times 3)\,\mathbf{i} + 2^2\,\mathbf{j} + 2\times 2\,\mathbf{k} = -2\,\mathbf{i} + 4\,\mathbf{j} + 4\,\mathbf{k}$.

Solution to Question 20

If $\mathbf{F} = F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k}$, $\quad$ then $\boldsymbol{\nabla}\cdot\mathbf{F} = \operatorname{div}\mathbf{F} = \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_3}{\partial z}$

and

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{F} &= \mathbf{curl}\,\mathbf{F}\\ &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ F_1 & F_2 & F_3\end{vmatrix}\\ &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.\end{aligned}$$

Here, $\mathbf{F} = x^2y\,\mathbf{i} - 2xz\,\mathbf{j} + 2yz\,\mathbf{k}$, so,

$$\boldsymbol{\nabla}\cdot\mathbf{F} = \operatorname{div}\mathbf{F} = 2xy - 0 + 2y = 2y(x+1)$$

and

$$\begin{aligned}\boldsymbol{\nabla} \times \mathbf{F} &= \mathbf{curl}\,\mathbf{F}\\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & -2xz & 2yz \end{vmatrix}\\ &= (2z+2x)\,\mathbf{i} - 0\,\mathbf{j} + (-2z - x^2)\,\mathbf{k} = 2(x+z)\,\mathbf{i} - (x^2+2z)\,\mathbf{k}.\end{aligned}$$

At $(1, 2, -1)$,

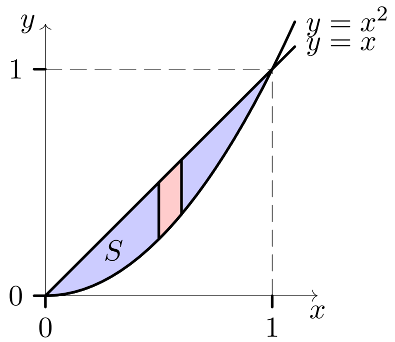
$$\boldsymbol{\nabla} \cdot \mathbf{F} = \operatorname{div} \mathbf{F} = 2 \times 2(1+1) = 8$$

and

$$\boldsymbol{\nabla} \times \mathbf{F} = \mathbf{curl}\,\mathbf{F} = 2(1-1)\,\mathbf{i} - (1-2)\,\mathbf{k} = \mathbf{k}.$$

Solution to Question 21

First sketch the region of integration, S:



Note that $y = x$ and $y = x^2$ intersect where $x = 0$ and $x = 1$, giving the minimum and maximum values of x for the region S.

In Cartesian coordinates we can write $\displaystyle\int_S xy\,dA = \int\!\!\int_S xy\,dy\,dx,$

in which we integrate with respect to y first, and then respect to x, and hence divide the region of integration into strips as shown in the diagram above.

Using the diagram to identify the limits of x and y we obtain

$$\int_S xy\,dA = \int_{x=0}^{x=1} \left(\int_{y=x^2}^{y=x} xy\,dy \right) dx.$$

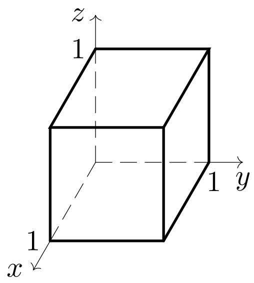
Now, $\displaystyle\int_{y=x^2}^{y=x} xy\,dy = x\left[\tfrac{1}{2}y^2\right]_{y=x^2}^{y=x} = x\left(\frac{1}{2}x^2 - \frac{1}{2}x^4\right) = \frac{1}{2}(x^3 - x^5),$

and hence

$$\int_S xy\,dA = \int_{x=0}^{x=1} \tfrac{1}{2}(x^3 - x^5)\,dx = \frac{1}{2}\left[\frac{1}{4}x^4 - \frac{1}{6}x^6\right]_0^1 = \frac{1}{2}\left(\frac{1}{4} - \frac{1}{6}\right) = \frac{1}{24}.$$

Solution to Question 22

First sketch the region of integration, B:



In Cartesian coordinates,

$$\int_B \left(x^2y + 2xz^2\right)\, dV = \int\!\!\int\!\!\int_B \left(x^2y + 2xz^2\right)\, dx\, dy\, dz.$$

Since the boundaries of the volume of integration are coordinate planes, we have

$$\begin{aligned}
\int_B \left(x^2y + 2xz^2)\right) dV &= \int_0^1\!\int_0^1\!\int_0^1 \left(x^2y + 2xz^2\right)\, dx\, dy\, dz \\
&= \int_0^1\!\int_0^1 \left[\frac{1}{3}x^3y + x^2z^2\right]_{x=0}^{x=1} dy\, dz \\
&= \int_0^1\!\int_0^1 \left(\frac{1}{3}y + z^2\right) dy\, dz \\
&= \int_0^1 \left[\frac{1}{6}y^2 + z^2y\right]_{y=0}^{y=1} dz \\
&= \int_0^1 \left(\frac{1}{6} + z^2\right) dz \\
&= \left[\frac{1}{6}z + \frac{1}{3}z^3\right]_{z=0}^{z=1} dz = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}.
\end{aligned}$$

Solution to Question 23

(a) The velocity, $\mathbf{v}$, is given by

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = -6\sin(3t)\,\mathbf{i} + 6\cos(3t)\,\mathbf{j} + 2\,\mathbf{k}.$$

(b) The speed is the magnitude of the velocity vector, so

$$|\mathbf{v}| = |\dot{\mathbf{r}}| = \sqrt{36\sin^2(3t) + 36\cos^2(3t) + 4} = \sqrt{40} = 2\sqrt{10}.$$

(c) The acceleration, $\mathbf{a}$, is given by

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \ddot{\mathbf{r}} = -18\cos(3t)\,\mathbf{i} - 18\sin(3t)\,\mathbf{j}$$

(d) The velocity vector points in the direction of the tangent to the trajectory, so the required unit vector is

$$\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{10}}(-3\sin(3t)\,\mathbf{i} + 3\cos(3t)\,\mathbf{j} + \mathbf{k}).$$

Solution to Question 24

(a) The force, $\mathbf{F}$, is given by Newton's second law: $\mathbf{F} = m\mathbf{a}$, where m is the mass of the particle.

So, $\mathbf{F} = 5(3\sin 6t\,\mathbf{i} + 3\cos 6t\,\mathbf{j} + 4\,\mathbf{k}) = 15\sin 6t\,\mathbf{i} + 15\cos 6t\,\mathbf{j} + 20\,\mathbf{k}.$.

Hence, the magnitude of the force,

$$|\mathbf{F}| = \sqrt{225\sin^2 6t + 225\cos^2 6t + 400} = \sqrt{225 + 400} = \sqrt{625} = 25.$$

(b) The velocity, $\mathbf{v}$, is given by

$$\begin{aligned}\mathbf{v} = \int \mathbf{a}\,dt &= \int (3\sin 6t\,\mathbf{i} + 3\cos 6t\,\mathbf{j} + 4\,\mathbf{k})\,dt \\ &= -\frac{1}{2}\cos 6t\,\mathbf{i} + \frac{1}{2}\sin 6t\,\mathbf{j} + 4t\,\mathbf{k} + \mathbf{c},\end{aligned}$$

where $\mathbf{c}$ is an arbitrary vector constant.

Since $\mathbf{v}(0) = \mathbf{i}$, we have $\quad -\frac{1}{2}\mathbf{i} + \mathbf{c} = \mathbf{i}, \quad$ so $\mathbf{c} = \frac{3}{2}\mathbf{i}$ and

$$\mathbf{v} = \left(\frac{3}{2} - \frac{1}{2}\cos 6t\right)\mathbf{i} + \frac{1}{2}\sin 6t\,\mathbf{j} + 4t\,\mathbf{k}.$$

The position, $\mathbf{r}$, is given by

$$\begin{aligned}\mathbf{r} = \int \mathbf{v}\,dt &= \int \left(\left(\frac{3}{2} - \frac{1}{2}\cos 6t\right)\mathbf{i} + \frac{1}{2}\sin 6t\,\mathbf{j} + 4t\,\mathbf{k}\right) dt \\ &= \left(\frac{3}{2}t - \frac{1}{12}\sin 6t\right)\mathbf{i} - \frac{1}{12}\cos 6t\,\mathbf{j} + 2t^2\,\mathbf{k} + \mathbf{d},\end{aligned}$$

where $\mathbf{d}$ is an arbitrary vector constant.

Since $\mathbf{r}(0) = \mathbf{j} + \mathbf{k}$, we have $\quad -\frac{1}{12}\mathbf{j} + \mathbf{d} = \mathbf{j} + \mathbf{k}, \quad$ so $\mathbf{d} = \frac{13}{12}\mathbf{j} + \mathbf{k}$ and

$$\mathbf{r} = \left(\frac{3}{2}t - \frac{1}{12}\sin 6t\right)\mathbf{i} + \left(\frac{13}{12} - \frac{1}{12}\cos 6t\right)\mathbf{j} + \left(1 + 2t^2\right)\mathbf{k}.$$

Solution to Question 25

(a) The solution of the differential equation is $\quad x = A\cos 2t + B\sin 2t,$

for arbitrary constants A and B.

When $t = 0$, $x = 3$, so $\quad 3 = A$ and $\quad x = 3\cos 2t + B\sin 2t.$.

Hence $x'(t) = -6\sin 2t + 2B\cos 2t$, and since $x'(0) = 0$, then $2B = 0$ and so $B = 0$.

So the motion is given by $\quad x = 3\cos 2t.$

The particle is at $x = 0$ when $3\cos 2t = 0$, that is $\cos 2t = 0$.

The first solution for $t > 0$ is $2t = \pi/2$, that is $t = \pi/4$.

We know $x'(t) = -6\sin 2t$, so at $t = \pi/4$: $\quad x'(t) = -6\sin\frac{\pi}{2} = -6.$

So the speed of the particle as it crosses $x = 0$ for the first time is 6.

(b) The angular frequency, $\omega = \sqrt{4} = 2$. So the period is given by $T = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi.$

Solution to Question 26

(a) $(2+i)+2(3-4i) = 2+i+6-8i = 8-7i$.

(b) $(2+i)(3-4i) = 6-8i+3i-4i^2 = 6-5i+4 = 10-5i$.

(c) $\dfrac{w}{z} = \dfrac{2+i}{3-4i}$. This can be expressed in Cartesian form by rationalising the denominator.

$$\frac{2+i}{3-4i} = \frac{2+i}{3-4i} \times \frac{3+4i}{3+4i} = \frac{6+8i+3i+4i^2}{9-16i^2} = \frac{2+11i}{25} = \frac{2}{25} + \frac{11}{25}i.$$

Solution to Question 27

The Fourier series for a function $f(x)$ over $-L < x < L$ is given by

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \qquad \text{and} \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx.$$

Here, $f(x) = 1+x$ and $L = 1$, so

$$a_0 = \int_{-1}^{1}(1+x)\,dx = \left[x + \frac{x^2}{2}\right]_{-1}^{1} = 1 + \frac{1}{2} - \left(-1 + \frac{1}{2}\right) = 2$$

$$\begin{aligned}
a_n &= \int_{-1}^{1}(1+x)\cos n\pi x\,dx \\
&= \left[(1+x)\frac{1}{n\pi}\sin n\pi x\right]_{-1}^{1} - \int_{-1}^{1}\left(1 \times \frac{1}{n\pi}\sin n\pi x\right)\,dx \\
&= 0 + \frac{1}{n^2\pi^2}\left[\cos n\pi x\right]_{-1}^{1} = \frac{1}{n^2\pi^2}\left(\cos n\pi - \cos(-n\pi)\right) = 0
\end{aligned}$$

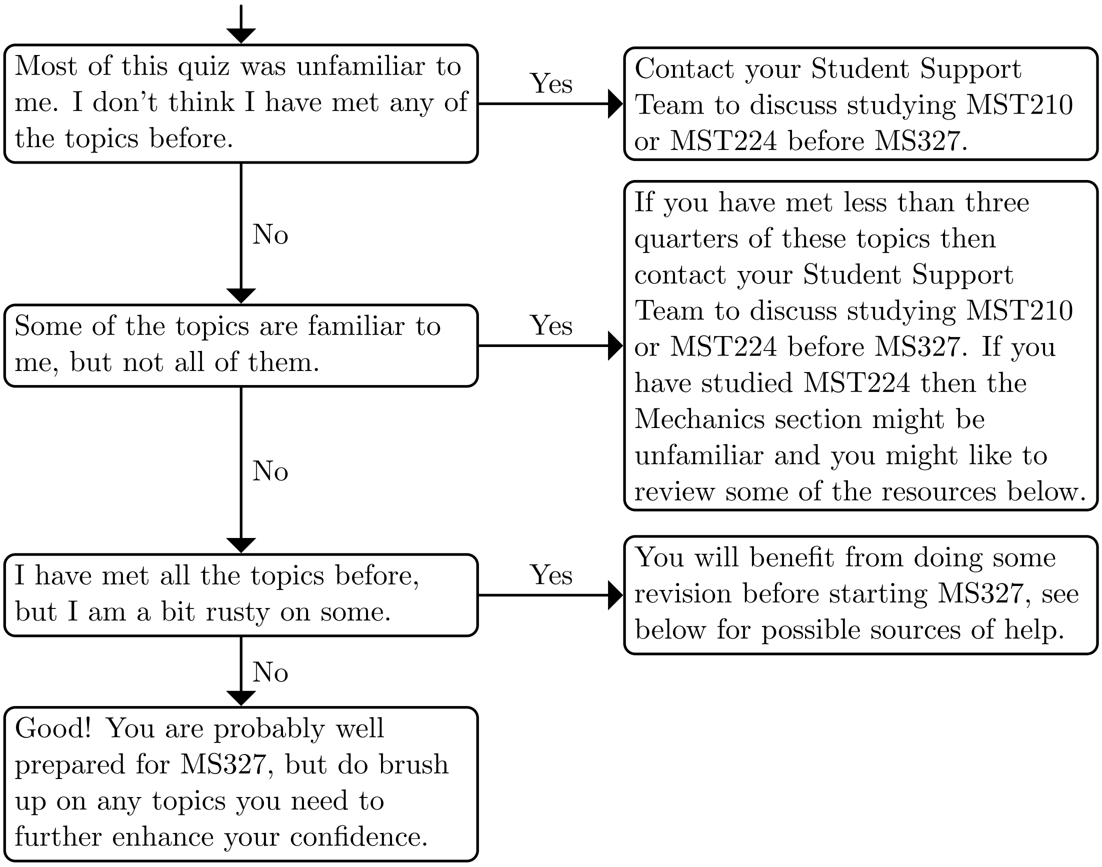
$$\begin{aligned}
b_n &= \int_{-1}^{1}(1+x)\sin n\pi x\,dx \\
&= \left[-(1+x)\frac{1}{n\pi}\cos n\pi x\right]_{-1}^{1} + \int_{-1}^{1}\left(1 \times \frac{1}{n\pi}\cos n\pi x\right)\,dx \\
&= -\frac{2}{n\pi}\cos n\pi + \frac{1}{n^2\pi^2}\left[\sin n\pi x\right]_{-1}^{1} \\
&= -\frac{2}{n\pi}\cos n\pi + \frac{1}{n^2\pi^2}\left(\sin n\pi - \sin(-n\pi)\right) \\
&= -\frac{2}{n\pi}\cos n\pi = -\frac{2}{n\pi}(-1)^n
\end{aligned}$$

Hence, the Fourier series is

$$F(x) = 1 - \sum_{n=1}^{\infty}\frac{2}{n\pi}(-1)^n \sin\frac{n\pi x}{L}.$$

What can I do to prepare for MS327?

Now that you have finished the quiz, how did you get on? The information below should help you to decide what, if anything, you should do next.



If you have any queries contact your Student Support Team via StudentHome.

What resources are there to help me prepare for MS327?

If you have studied MST210, its predecessor MST209, or MST224, then you could use parts of these to revise for MS327.

If you need to brush up some of the more basic topics like algebra, trigonometry and calculus, then you may find revising material from MST124 and MST125 (or their predecessors MST121 and MS221) helpful. Alternatively, such material is often covered by standard A-level textbooks.

If you have studied MST224 rather than MST210 then you might not have met mechanics before (see the Mechanics section of the diagnostic quiz). You might like to review some material on this, such as the MST210 Bridging material available at:

mcs-notes2.open.ac.uk/WebResources/Maths.nsf/A/odm8/$FILE/mst210bridging.pdf,

or a A-level mechanics textbook.

The mathcentre web-site (`www.mathcentre.ac.uk`) includes several teach-yourself books, summary sheets, revision booklets, online exercises and video tutorials on a range of mathematical skills.