Faculty of Science, Technology, Engineering and Mathematics
MST374 Computational applied mathematics

MST374 Diagnostic quiz

Are you ready for MST374?

This diagnostic quiz is designed to help you decide if you are ready to study *Computational applied mathematics* (MST374). This document also contains some advice on preparatory work that you may find useful before starting MST374 (see below and page 6). The better prepared you are for MST374 the greater your chance of success.

The topics which are included in this quiz are those that we expect you to be familiar with before you start the module. If you have previously studied *Mathematical methods and modelling* (MST210) or *Mathematical methods* (MST224), you should be familiar with most of the topics covered in the quiz.

If you can complete the quiz with only the occasional need to look at other material, then you should be well prepared for MST374. If you find some topics that you remember having met before but need to do some more work on, then you should consider the suggestions for refreshing your knowledge on page 6.

Try the questions now, and then see the notes on page 6 of this document to see if you are ready for MST374.

Do contact your Student Support Team via StudentHome if you have any queries about MST374, or your readiness to study it.

Tables of standard derivatives and integrals

Function	Derivative
a	0
x^a	ax^{a-1}
e^{ax}	ae^{ax}
$\ln(ax)$	$\frac{1}{x}$
$\sin(ax)$	$a\cos(ax)$
$\cos(ax)$	$-a\sin(ax)$
$\tan(ax)$	$a\sec^2(ax)$
$\cot(ax)$	$-a\,\mathrm{cosec}^2(ax)$
$\sec(ax)$	$a\sec(ax)\tan(ax)$
$\mathrm{cosec}(ax)$	$-a\,\mathrm{cosec}(ax)\cot(ax)$
$\arcsin(ax)$	$\frac{a}{\sqrt{1-a^2x^2}}$
$\arccos(ax)$	$-\frac{a}{\sqrt{1-a^2x^2}}$
$\arctan(ax)$	$\frac{a}{1+a^2x^2}$
$\mathrm{arccot}(ax)$	$-\frac{a}{1+a^2x^2}$
$\mathrm{arcsec}(ax)$	$\frac{a}{\lvert ax\rvert\sqrt{a^2x^2-1}}$
$\mathrm{arccosec}(ax)$	$-\frac{a}{\lvert ax\rvert\sqrt{a^2x^2-1}}$

Function	Integral
a	ax
$x^a \ (a \neq -1)$	$\frac{x^{a+1}}{a+1}$
$\frac{1}{ax+b}$	$\frac{1}{a}\ln\lvert ax+b\rvert$
e^{ax}	$\frac{1}{a}e^{ax}$
$\ln(ax)$	$x(\ln(ax)-1)$
$\sin(ax)$	$-\frac{1}{a}\cos(ax)$
$\cos(ax)$	$\frac{1}{a}\sin(ax)$
$\tan(ax)$	$-\frac{1}{a}\ln\lvert\cos(ax)\rvert$
$\cot(ax)$	$\frac{1}{a}\ln\lvert\sin(ax)\rvert$
$\sec(ax)$	$\frac{1}{a}\ln\lvert\sec(ax)+\tan(ax)\rvert$
$\mathrm{cosec}(ax)$	$\frac{1}{a}\ln\lvert\mathrm{cosec}(ax)-\cot(ax)\rvert$
$\sec^2(ax)$	$\frac{1}{a}\tan(ax)$
$\mathrm{cosec}^2(ax)$	$-\frac{1}{a}\cot(ax)$
$\frac{1}{x^2+a^2}$	$\frac{1}{a}\arctan\left(\frac{x}{a}\right)$
$\frac{1}{(x-a)(x-b)}$	$\frac{1}{a-b}\ln\left\lvert\frac{a-x}{x-b}\right\rvert$
$\frac{1}{\sqrt{x^2+a^2}}$	$\ln(x+\sqrt{x^2+a^2}\,)$ *or* $\mathrm{arcsinh}\left(\frac{x}{a}\right)$
$\frac{1}{\sqrt{x^2-a^2}}$	$\ln\lvert x+\sqrt{x^2-a^2}\,\rvert$ *or* $\mathrm{arccosh}\left(\frac{x}{a}\right)$
$\frac{1}{\sqrt{a^2-x^2}}$	$\arcsin\left(\frac{x}{a}\right)$

Diagnostic quiz questions

Vectors and matrices

Question 1

Consider the vectors $\mathbf{a} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$. Find

(a) $\mathbf{a} + \mathbf{b}$ (b) $2\mathbf{a} + 3\mathbf{b}$ (c) $|\mathbf{b}|$ (d) $\mathbf{a} \cdot \mathbf{b}$

Question 2

If $\mathbf{x} = (3 \ \ 1 \ \ -2 \ \ 4)^T$ and $\mathbf{y} = (-2 \ \ 5 \ \ 1 \ \ 3)^T$, calculate each of the following:

(a) $2\mathbf{x} + 3\mathbf{y}$, (b) $|\mathbf{x}|$, (c) $\mathbf{x}^T\mathbf{y}$.

Question 3

Find the vector equation of the line segment between the points with position vectors $\mathbf{p} = (2 \ \ -3 \ \ 1)^T$ and $\mathbf{q} = (5 \ \ -2 \ \ -3)^T$.

Question 4

Let $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 & -1 \\ 2 & 5 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} 5 & 4 \\ 2 & 0 \\ 1 & 3 \end{pmatrix}$.

Calculate each of the following (or state why the calculation is not possible):

(a) $2\mathbf{A} + \mathbf{B}$ (b) $\mathbf{A} - 3\mathbf{C}$ (c) $\mathbf{AB}$

(d) $\mathbf{AC}$ (e) $\mathbf{CA}$ (f) $\mathbf{BC}^T$

Question 5

Calculate the determinants of the following matrices:

(a) $\begin{pmatrix} 2 & 4 \\ -3 & 5 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 1 & 0 \\ 2 & -3 & 2 \\ 4 & 2 & -1 \end{pmatrix}$

Question 6

Calculate the inverse of the matrix $\begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix}$.

Question 7

Find the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 1 & -1 \\ -4 & -2 \end{pmatrix}$.

System of equations

Question 8

Solve, if possible, the following systems of equations:

(a) $2x + 3y + z = 5$
$x - 4y + 2z = 11$
$3x + y - 3z = 2$

(b) $x - 5y + 2z = 4$
$2x + y + 3z = -2$
$4x - 9y + 7z = 0$

(c) $x + 3y - 2z = 1$
$2x - y + 3z = 2$
$4x + 5y - z = 4$

Differentiation

Question 9

Differentiate the following functions with respect to x.

(a) $f(x) = 3x^4 + 2\cos x - 4\ln x$

(b) $g(x) = \sin 3x + e^{2x}$

(c) $h(x) = x^5 \tan 2x$

(d) $p(x) = \dfrac{2x^3 + 3x + 1}{2x + 1}$

Question 10

Find the second derivative with respect to t of $f(t) = 2t^5 + 3\ln(t)$.

Question 11

Find and classify the stationary points of $y(x) = x^3 - 3x^2 - 9x + 6$.

Ordinary differential equations

Question 12

Find the general solution of the following differential equation.

$$\frac{dy}{dx} = x^2 - \frac{2y}{x} \quad (x > 0).$$

Question 13

Solve each of the following initial-value problems.

(a) $\dfrac{dy}{dx} = \dfrac{xy}{x^2 + 1}, \quad y(0) = 1.$ (b) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - 3y = 0, \quad y(0) = 0,\ y'(0) = 4.$

Question 14

Solve the following boundary-value problem.

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 3 + 5x, \quad y'(0) = 1,\ y(\pi) = 0.$$

Partial differentiation

Question 15

Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ when $f(x,y) = 3x^2 \sin y + \log(x^2 y^4)$.

Question 16

Find all the second partial derivatives of $g(x,y) = 2x^4 \cos(x+2y)$.

Question 17

Locate and classify all the stationary points of the function $f(x,y) = 2x^3 + y^3 - 6x - 12y$.

Vector calculus

Question 18

If $f(x,y,z) = x^2 z + yz^2 + xy^2$, find $\boldsymbol{\nabla} f$ (also known as $\mathbf{grad} f$) at the point $(1,3,-2)$.

Hence find the rate of change of f in the direction $(1 \;\; 2 \;\; -2)^T$ at the point $(1,3,-2)$.

Taylor series

Question 19

Calculate the first three terms of the Taylor series about $\pi/2$ for the function $f(x) = \sin 2x$.

What can I do to prepare for MST374?

Now that you have finished the quiz, how did you get on? The information below should help you to decide what to do next.

Most of this quiz was unfamiliar to me. I don't think I have met any of the topics before.	You should consider studying MST210 or MST224 before MST374. Contact your Student Support Team to discuss this.
Some of these topics are familiar to me, but not all of them.	If you have met less than 75% of these topics then you should consider studying MST210 or MST224 before MST374. Contact your Student Support Team to discuss this.
I have met all these topics before, but I am a bit rusty on some, and had to look up how to do a few of them.	You will benefit from doing some revision before starting MST374. See below for possible sources of help.
I can do all these questions.	You are probably well prepared for MST374, but do brush up on any topics you need to further enhance your confidence.

If you have any queries contact your Student Support Team via StudentHome.

What resources are there to help me prepare for MST374?

If you have studied MST210, its predecessor MST209, or MST224, then you could use parts of these to revise for MST374.

If you need to do some background preparation before starting MST374, we suggest you concentrate your efforts on the following topics.

(1) Linear algebra (vectors and matrices)

(2) Differentiation (both for functions of one variable and functions of several variables)

(3) Simple ordinary differential equations.

(4) Vector calculus (particularly the gradient of a function of several variables).

If you need to brush up some of the more basic topics like algebra, trigonometry and calculus, then you may find revising material from MST124 and MST125 (or their predecessors MST121 and MS221) helpful. Alternatively, such material is often covered by standard A-level textbooks.

The mathcentre web-site (`www.mathcentre.ac.uk`) includes several teach-yourself books, summary sheets, revision booklets, online exercises and video tutorials on a range of mathematical skills.

Solutions to questions

Solution to Question 1

(a) $\mathbf{a}+\mathbf{b}=\begin{pmatrix}2\\5\end{pmatrix}+\begin{pmatrix}1\\-3\end{pmatrix}=\begin{pmatrix}3\\2\end{pmatrix}.$

(b) $2\mathbf{a}+3\mathbf{b}=2\begin{pmatrix}2\\5\end{pmatrix}+3\begin{pmatrix}1\\-3\end{pmatrix}=\begin{pmatrix}4\\10\end{pmatrix}+\begin{pmatrix}3\\-9\end{pmatrix}=\begin{pmatrix}7\\1\end{pmatrix}.$

(c) $|\mathbf{b}|=\sqrt{1^2+(-3)^2}=\sqrt{10}.$

(d) $\mathbf{a}\cdot\mathbf{b}=\begin{pmatrix}2\\5\end{pmatrix}\cdot\begin{pmatrix}1\\-3\end{pmatrix}=2\times 1+5\times(-3)=2-15=-13.$

Solution to Question 2

(a)
$$\begin{aligned}2\mathbf{x}+3\mathbf{y}&=2(3,1,-2,4)^T+3(-2,5,1,3)^T\\&=(6,2,-4,8)^T+(-6,15,3,9)^T\\&=(0,17,-1,17)^T.\end{aligned}$$

(b) $|\mathbf{x}|=\sqrt{3^2+1^2+(-2)^2+4^2}=\sqrt{30}.$

(c)
$$\mathbf{x}^T\mathbf{y}=(3,1,-2,4)\begin{pmatrix}-2\\5\\1\\3\end{pmatrix}$$
$$=3\times(-2)+1\times 5+(-2)\times 1+4\times 3=9.$$

Note that $\mathbf{x}^T\mathbf{y}=\mathbf{x}\cdot\mathbf{y}$.

Solution to Question 3

The vector equation of a line is of the form $\mathbf{x}=\mathbf{p}+t\mathbf{d}$ where $\mathbf{d}$ is a vector in the direction of a line.

In particular, the vector equation of a line between the points with position vectors $\mathbf{p}$ and $\mathbf{q}$ is $\mathbf{x}=\mathbf{p}+t(\mathbf{q}-\mathbf{p})$, where $\mathbf{p}$ is the point corresponding to $t=0$ and $\mathbf{q}$ the point corresponding to $t=1$.

Here, $\mathbf{q}-\mathbf{p}=(5,-2,-3)^T-(2,-3,1)^T=(3,1,-4)^T$.

So the required line segment is

$$\mathbf{x}=(2,-3,1)^T+t(3,1,-4)^T,\qquad 0\le t\le 1$$

or, equivalently

$$\mathbf{x}=(2+3t,-3+t,1-4t)^T,\qquad 0\le t\le 1.$$

Solution to Question 4

(a) $2\mathbf{A}+\mathbf{B}=2\begin{pmatrix}2&3\\-1&4\end{pmatrix}+\begin{pmatrix}3&-1\\2&5\end{pmatrix}=\begin{pmatrix}4&6\\-2&8\end{pmatrix}+\begin{pmatrix}3&-1\\2&5\end{pmatrix}=\begin{pmatrix}7&5\\0&13\end{pmatrix}.$

(b) $\mathbf{A}-3\mathbf{C}$

This cannot be calculated since $\mathbf{A}$ has size 2×2 and $\mathbf{C}$ (hence $3\mathbf{C}$) has size 3×2, which differ.

(c) $\mathbf{AB}=\begin{pmatrix}2&3\\-1&4\end{pmatrix}\begin{pmatrix}3&-1\\2&5\end{pmatrix}=$
$$\begin{pmatrix}2\times 3+3\times 2 & 2\times(-1)+3\times 5\\(-1)\times 3+4\times 2 & (-1)\times(-1)+4\times 5\end{pmatrix}=\begin{pmatrix}12&13\\5&21\end{pmatrix}.$$

(d) $\mathbf{A}$ has size 2×2 and $\mathbf{C}$ has size 3×2. Since the number of columns of $\mathbf{A}$ is different to the number of rows of $\mathbf{C}$, $\mathbf{AC}$ cannot be calculated.

(e) $\mathbf{CA} = \begin{pmatrix} 5 & 4 \\ 2 & 0 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 6 & 31 \\ 4 & 6 \\ -1 & 15 \end{pmatrix}$.

(f) $\mathbf{BC}^T = \begin{pmatrix} 3 & -1 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} 5 & 2 & 1 \\ 4 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 11 & 6 & 0 \\ 30 & 4 & 17 \end{pmatrix}$.

Solution to Question 5

(a) $\begin{vmatrix} 2 & 4 \\ -3 & 5 \end{vmatrix} = 2 \times 5 - 4 \times (-3) = 22.$

(b)
$$\begin{vmatrix} 3 & 1 & 0 \\ 2 & -3 & 2 \\ 4 & 2 & -1 \end{vmatrix} = 3 \times \begin{vmatrix} -3 & 2 \\ 2 & -1 \end{vmatrix} - 1 \times \begin{vmatrix} 2 & 2 \\ 4 & -1 \end{vmatrix} + 0 \times \begin{vmatrix} 2 & -3 \\ 4 & 2 \end{vmatrix}$$
$$= 3\left((-3) \times (-1) - 2 \times 2\right) - 1 \times \left(2 \times (-1) - 2 \times 4\right)$$
$$= 3 \times (-1) - (-10) = 7.$$

Solution to Question 6

We have $\begin{vmatrix} 2 & -1 \\ -4 & 3 \end{vmatrix} = 2$. Therefore, $\begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ 2 & 1 \end{pmatrix}$.

Solution to Question 7

The characteristic equation is given by $\begin{vmatrix} 1-\lambda & -1 \\ -4 & -2-\lambda \end{vmatrix} = 0$,

that is, $(1-\lambda)(-2-\lambda) - 4 = 0$ or, $\lambda^2 + \lambda - 6 = 0$.

This can be factorised as $(\lambda + 3)(\lambda - 2) = 0$, hence the eigenvalues are 2 and -3.

The eigenvector associated with the eigenvalue λ are found by solving

$$\begin{pmatrix} 1-\lambda & -1 \\ -4 & -2-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0.$$

For $\lambda = 2$ this gives

$$\begin{pmatrix} -1 & -1 \\ -4 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

which can be written as

$$-x - y = 0$$
$$-4x - 4y = 0.$$

These are satisfied when $x = -y$ hence an eigenvector is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

For $\lambda = -3$ the equation gives

$$\begin{pmatrix} 4 & -1 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

which can be written as

$$\begin{aligned} 4x - y &= 0 \\ -4x + y &= 0. \end{aligned}$$

These are satisfied when $x = y/4$ hence an eigenvector is $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$.

Solution to Question 8

(a) The system is

$$\begin{aligned} 2x+3y+\ z &= 5 && \text{(S1)} \\ x-4y+2z &= 11 && \text{(S2)} \\ 3x+\ y-3z &= 2. && \text{(S3)} \end{aligned}$$

Eliminate z from equations (S2) and (S3):

$$\begin{aligned} & 2x+\ 3y+z = 5 && \text{(S4)} \\ (S2) - 2 \times (S1): \quad & -3x-10y \quad = 1 && \text{(S5)} \\ (S3) + 3 \times (S1): \quad & 9x+10y \quad = 17 && \text{(S6)} \end{aligned}$$

Eliminate y from equation (S6).

$$\begin{aligned} & 2x+\ 3y+z = 5 && \text{(S7)} \\ & -3x-10y \quad = 1 && \text{(S8)} \\ (S5) + (S6): \quad & 6x \qquad\quad = 18 && \text{(S9)} \end{aligned}$$

So, from (S9), $x = 3$, then using (S8), $y = -1$ and from (S7), $z = 2$.

(b) The system is

$$\begin{aligned} x-5y+2z &= 4 && \text{(S10)} \\ 2x+\ y+3z &= -2 && \text{(S11)} \\ 4x-9y+7z &= 0. && \text{(S12)} \end{aligned}$$

Eliminate x from equations (S11) and (S12).

$$\begin{aligned} & x-\ 5y+2z = 4 && \text{(S13)} \\ (S11) - 2 \times (S10): \quad & 11y-\ z = -10 && \text{(S14)} \\ (S12) - 4 \times (S10): \quad & 11y-\ z = -16 && \text{(S15)} \end{aligned}$$

Equations (S14) and (S15) are inconsistent, so the system has no solution.

(c) The system is

$$\begin{aligned} x+3y-2z &= 1 && \text{(S16)}\\ 2x-\ y+3z &= 2 && \text{(S17)}\\ 4x+5y-\ z &= 4. && \text{(S18)} \end{aligned}$$

Eliminate x from equations (S17) and (S18).

$$\begin{aligned} & x+3y-2z = 1 && \text{(S19)}\\ (S17) - 2\times(S16):\quad & -7y+7z = 0 && \text{(S20)}\\ (S18) - 4\times(S16):\quad & -7y+7z = 0 && \text{(S21)} \end{aligned}$$

Equations (S20) and (S21) are identical, so there is one degree of freedom. Let $z = c$, a constant, then to satisfy (S20) and (S21), $y = c$. From equation (S19), $x = 1 - c$.

So the family of solutions satisfying the system is
$x = 1 - c$, $y = c$, $z = c$.

Solution to Question 9

(a) $f'(x) = 12x^3 - 2\sin x - \dfrac{4}{x}$

(b) $g'(x) = 3\cos 3x + 2e^{2x}$

(c) $h'(x) = 5x^4 \tan 2x + 2x^5 \sec^2 2x$

(d)
$$\begin{aligned} p'(x) &= \frac{(2x+1)(6x^2+3) - (2x^3+3x+1)(2)}{(2x+1)^2}\\ &= \frac{12x^3+6x+6x^2+3-4x^3-6x-2}{(2x+1)^2}\\ &= \frac{8x^3+6x^2+1}{(2x+1)^2} \end{aligned}$$

Solution to Question 10

$$f'(t) = 10t^4 + \frac{3}{t} = 10t^4 + 3t^{-1}, \quad \text{so } f''(t) = 40t^3 - 3t^{-2}.$$

Solution to Question 11

Stationary points occur when $\dfrac{dy}{dx} = 0$.

Here, $\dfrac{dy}{dx} = 3x^2 - 6x - 9$, so we need to solve $3x^2 - 6x - 9 = 0$, or $x^2 - 2x - 3 = 0$.

Factorising gives $(x-3)(x+1) = 0$, so the stationary points occur at $x = -1$ and $x = 3$.

The nature of the points can be classified using the second derivative test.

Here, $\dfrac{d^2y}{dx^2} = 6x - 6$.

So, at $x = -1$, $\dfrac{d^2y}{dx^2} = -12 < 0$ hence point is a local maximum.

At $x = 3$, $\dfrac{d^2y}{dx^2} = 12 > 0$ hence point is a local minimum.

Solution to Question 12

This equation can be solved by using the integrating factor method. The integrating factor is $p(x) = \exp\int \frac{2}{x}dx = x^2$ and we can write $\frac{d}{dx}(x^2 y) = x^4$. We solve this equation by direct integration to find the general solution $y(x) = \frac{x^3}{5} + \frac{C}{x^2}$ where C is an arbitrary constant.

Solution to Question 13

(a) We use separation of variables and write:

$$\frac{1}{y}\frac{dy}{dx} = \frac{x}{x^2+1}.$$

Integrating both sides we obtain $\ln y = \frac{1}{2}\ln(x^2+1) + C$, where C is an arbitrary constant. Hence

$$y(x) = A\sqrt{x^2+1},$$

where we have defined $A = e^C$. Imposing the condition $y(0) = A = 1$. Hence, $y(x) = \sqrt{x^2+1}$.

(b) This is a homogeneous linear second-order differential equation with constant coefficients. The auxiliary equation is $\lambda^2 + 2\lambda - 3 = 0$ which has roots $\lambda_1 = 1$ and $\lambda_2 = -3$. Therefore, the general solution is $y(x) = Ae^x + Be^{-3x}$, where A and B are arbitrary constants.

Imposing the given conditions we find $A + B = 0$ and $A - 3B = 4$. From the first relation we have $A = -B$, and so we find $-4B = 4$ which gives $B = -1$ and $A = 1$. Therefore, the solution is $y(x) = e^x - e^{-3x}$.

Solution to Question 14

We first solve the associated homogeneous equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0.$$

The auxiliary equation is $\lambda^2 - 2\lambda + 5 = 0$ which has roots

$$\lambda = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i.$$

Therefore, the complimentary function is $y_c(x) = e^x\,(A\cos(2x) + B\sin(2x))$, where A and B are arbitrary constants.

Returning to the inhomogeneous equation,

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 3 + 5x,$$

the form of the right-hand side suggests a trial solution of the form $y_p = ax + b$, so $y_p' = a$ and $y_p'' = 0$. Substituting into the equation gives

$$0 - 2a + 5(ax+b) = 3 + 5x \quad \text{or} \quad 5ax + 5b - 2a = 5x + 3.$$

Comparing coefficients gives $a = b = 1$. Therefore the general solution, $y = y_c + y_p$, is

$$y(x) = e^x\,(A\cos(2x) + B\sin(2x)) + x + 1.$$

The derivative of the solution is

$$y'(x) = e^x\left[(A+2B)\cos(2x) + (B-2A)\sin(2x)\right] + 1.$$

The first condition gives $y'(0) = A + 2B + 1 = 1$, and so $A = -2B$. Hence,

$$y(x) = e^x B\left(-2\cos(2x) + \sin(2x)\right) + x + 1.$$

Imposing the second condition, $y(\pi) = e^\pi B(-2) + \pi + 1 = 0$, gives $B = \frac{1+\pi}{2}e^{-\pi}$, and so $A = -(1+\pi)e^{-\pi}$. The solution is hence

$$y(x) = \frac{e^{-\pi}(1+\pi)}{2}e^x\left[\sin(2x) - 2\cos(2x)\right] + x + 1.$$

Solution to Question 15

$$\frac{\partial f}{\partial x} = 6x\sin y + \frac{2}{x}, \qquad \text{and} \qquad \frac{\partial f}{\partial y} = 3x^2\cos y + \frac{4}{y}.$$

Solution to Question 16

The first partial derivatives are

$$\frac{\partial g}{\partial x} = 8x^3\cos(x+2y) - 2x^4\sin(x+2y) \qquad \text{and} \qquad \frac{\partial g}{\partial y} = -4x^4\sin(x+2y).$$

The second partial derivatives are

$$\begin{aligned}
\frac{\partial^2 g}{\partial x^2} &= 24x^2\cos(x+2y) - 16x^3\sin(x+2y) - 2x^4\cos(x+2y)\\
\frac{\partial^2 g}{\partial x\partial y} &= -16x^3\sin(x+2y) - 4x^4\cos(x+2y)\\
\frac{\partial^2 g}{\partial y^2} &= -8x^4\cos(x+2y)\\
\frac{\partial^2 g}{\partial y\partial x} &= -16x^3\sin(x+2y) - 4x^4\cos(x+2y).
\end{aligned}$$

Solution to Question 17

We have $f_x = 6x^2 - 6$ and $f_y = 3y^2 - 12$, so the stationary points are at the points (x, y) where $x^2 = 1$ and $y^2 = 4$, namely $(1,2),\ (1,-2),\ (-1,2),\ (-1,-2)$.

Also, $f_{xx} = 12x$, $f_{xy} = 0$ and $f_{yy} = 6y$. Thus we have $AC - B^2 = f_{xx}f_{yy} - f_{xy}^2 = 72xy$.

At $(1,2)$ and $(-1,-2)$, this gives $AC - B^2 = 144$. Since $A > 0$ at $(1,2)$, this is a local minimum; since $A < 0$ at $(-1,-2)$, this is a local maximum.

At $(1,-2)$ and $(-1,2)$, we have $AC - B^2 = -144$, and these are saddle points.

Solution to Question 18

$$\boldsymbol{\nabla} f = \left(\frac{\partial f}{\partial x}\ \ \frac{\partial f}{\partial y}\ \ \frac{\partial f}{\partial z}\right)^T = \left(2xz + y^2 \quad z^2 + 2xy \quad x^2 + 2yz\right)^T$$

So at $(1,3,-2)$, $\quad \boldsymbol{\nabla} f(1,3,-2) = (5\ \ 10\ \ -11)^T$

The rate of change of f in the direction $\hat{\mathbf{s}}$, where $\hat{\mathbf{s}}$ is a unit vector, is given by $\boldsymbol{\nabla} f \cdot \hat{\mathbf{s}}$.

Here, the direction $\mathbf{s} = (1\ \ 2\ \ -2)^T$ hence

$$\hat{\mathbf{s}} = \frac{1}{3}\times(1\ \ 2\ \ -2)^T = \left(\frac{1}{3}\ \ \frac{2}{3}\ \ -\frac{2}{3}\right)^T.$$

So the rate of change of f in the direction of $\mathbf{s}$ at $(1, 3, -2)$ is

$$\boldsymbol{\nabla} f \cdot \hat{\mathbf{s}} = (5 \;\; 10 \;\; -11)^T \cdot \left(\frac{1}{3} \;\; \frac{2}{3} \;\; -\frac{2}{3}\right)^T = \frac{47}{3}.$$

Solution to Question 19

The Taylor series about a for the function $f(x)$ is

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots .$$

For $f(x) = \sin 2x$ we have $f(\pi/2) = 0$ and,

$$\begin{aligned} f'(x) &= 2\cos 2x & f'(\pi/2) &= -2 \\ f''(x) &= -4\sin 2x & f''(\pi/2) &= 0 \\ f'''(x) &= -8\cos 2x & f'''(\pi/2) &= 8 \\ f^{(iv)}(x) &= 16\sin 2x & f^{(iv)}(\pi/2) &= 0 \\ f^{(v)}(x) &= 32\cos 2x & f^{(v)}(\pi/2) &= -32. \end{aligned}$$

So,

$$\begin{aligned} f(x) &= 0 - 2\left(x - \frac{\pi}{2}\right) + 0\left(x - \frac{\pi}{2}\right)^2 + \frac{8}{3!}\left(x - \frac{\pi}{2}\right)^3 + 0\left(x - \frac{\pi}{2}\right)^4 - \frac{32}{5!}\left(x - \frac{\pi}{2}\right)^5 + \cdots \\ &= -2\left(x - \frac{\pi}{2}\right) + \frac{8}{3!}\left(x - \frac{\pi}{2}\right)^3 - \frac{32}{5!}\left(x - \frac{\pi}{2}\right)^5 + \cdots \end{aligned}$$