1.2 Starting with examples

This section is designed to provoke algebraic thinking without using any symbols beyond ordinary number and operation signs. It is an example of the strand of algebraic thinking concerned with generalising. You will be asked to notice and express what is general about a collection of mathematical objects. We are dealing with algebra so these objects are calculations or, to be more precise, numerical expressions.

The statements you make will express your thinking from its early stages and because of this you can think of them as conjectures; statements that suggest a possible truth in mathematics. Describing something as a conjecture accepts that it needs further work.

Task 1.4 Times or subtract

Work out (with or without a calculator):

$$\begin{array}{ll} 32 \times 18 & 640 - 64 \\ 42 \times 18 & 840 - 84 \\ 52 \times 18 & 1040 - 104 \end{array}$$

What do you notice? What will you do next?

Save Reset

Discussion

In the previous section you were encouraged to think about relationships and about generality. Here we have six numerical expressions that can be formed into three pairs that each have the same value. We could write three equations to show the three relationships. We focus on generality by noticing what is the same about these three equations and wondering if there are any more.

You are going to return to this task as you go through the rest of the section.

Being a mathematician, you are probably curious about whether you could extend this idea to more examples or even a general case. At this point you may want to satisfy that curiosity and follow your own line of thought. If you do this before reading on, try to be aware of what you are thinking and saying and what drives each of your choices or decisions. Then you can compare the steps in your investigation to the approaches described in the rest of the section. Otherwise, just read on and follow each approach as you meet it.

1.2.1 Exemplifying

Task 1.5 Exemplifying

(a) Creating and reflecting on examples

Create one or two more example pairs that match what you have noticed.

Now reflect on what you have done. Did you have a method for creating more examples?

How did you decide what to write for the subtractions? Write down what you were noticing.

Save and reveal discussion Reset

(b) Recognising strategies

Do you recognise any of these descriptions of what people have noticed as their own key moments?

- I saw the repeated 6 and 4 before and after the minus sign …
- I saw a number ending in 0 then you subtract the same number without a 0 …
- I saw the pattern 6, 8, 10 always before a 4 …
- I saw that the 6 in 640 is twice the 3 in 32 and also the middle 4 is twice the 2 …
- I was thinking of how to multiply 32 by 18 and saw the first number was more than 10 times 32, so then I thought about 20 times …
- I saw that 104 was double 52 … the last was double the first.

These responses suggest several strategies for creating new examples:

- continuing a pattern suggested by moving down the column
- using relationships suggested by comparing different terms in the same row
- carrying out the multiplication and comparing stages in that process to what is written.

(c) Checking your answers

Having more than one way of getting to an answer can help you check that it is correct. How did you check your examples? Did you do the calculation for the multiplication and the subtraction separately and verify that each gave the same value? Or was there a reason that made you so sure it would work that you did not even do the calculations?

You may not have stayed with the near examples such as 12 or 62. If you haven't already done so, write the example pair that starts 92×18. And the one that starts 212×18.

Save and reveal discussion Reset

At this stage, let us review the approach so far. This task is concerned with generalising: noticing a general feature suggested by working with examples that were given to you. If the situation is relatively new to you, the general idea will not arrive fully formed in your mind. You need to have strategies to think about it and around it. You will need to explore how general it is, to consider what is and isn't included in your idea. This is called exploring generality.

One way of exploring is to create and organise new examples with the aim of deciding whether or not they fit your emerging sense of generality. This is called exemplifying. By staying with particular examples, you have the advantages of using familiar, precise number signs and of being able to check each example for accuracy. This is probably not yet algebraic thinking since when you create examples and check them you are mostly working with numbers. Your thinking moves towards the algebraic if

- you realise that you are being guided by awareness of pattern and structure
- you start comparing the examples for their similarities and differences
- you notice and articulate what you see as an overall or general method.

While generalising is natural, learners need time to notice that they have a sense of generality, to explore and talk about generality, to strengthen and extend their natural powers. One of the ways teachers can support learners in developing these abilities is to ask them to give an example and say why it is an example. This is a strategy that supports sense-making. Throughout the module you will be asked to notice your own reasoning and how it is algebraic. Sometimes you will notice that you have naturally started to create and organise examples, putting them together in ways that suggest generality. Other times you may be stuck and decide consciously to exemplify as a way of getting unstuck.

Some patterns of reasoning and pedagogic strategies are so central to algebraic thinking that you will need to recognise them time and again. These ideas are known as the module ideas. You have just been introduced to two of these ideas: **exploring generality**, and the linked idea of **exemplifying and generalising**.

1.2.2 Exploring and expressing generality

One aim of this task is for you to write, or say aloud, one or more general statements based on the original examples. A way of approaching this is to say anything that you have noticed is a shared feature. You get better at this through practice, so it does not matter how simply you start. Some simple general statements could be:

> Every example starts with a number that ends in two, which is multiplied by 18.

> Each subtraction sum starts with a multiple of ten.

Clearly these two statements do not tell us everything about this class of examples. Are they enough to create another example?

What would you think if I suggested $2 \times 18 = 50 - 14$? It is an example that fits these two statements but it does not have the appealing pattern of repeated digits.

It would be unusual for anyone to arrive straightaway at one statement that encapsulates exactly what their general idea is. However, simply writing down what you know is a useful strategy for anyone learning and doing algebra. Saying or writing a general statement is called *expressing a generalisation* or *expressing generality*. Putting your ideas into words is important. It reduces what you are trying to remember, gives the ideas a definite form and lets you start to work on them to reach a statement that describes more closely what you see as the generality.

Another way of encouraging yourself to express a generalisation is to try to explain to someone else what you have been doing. Let's apply this to the task.

Task 1.6 Expressing generality

How would you explain to someone how to create a new example of this general case? Try doing this in the following two ways:

- Using words only and not referring to a previous example
- Adding comments to this example number sentence:

$$32 \times 18 = 640 - 64$$

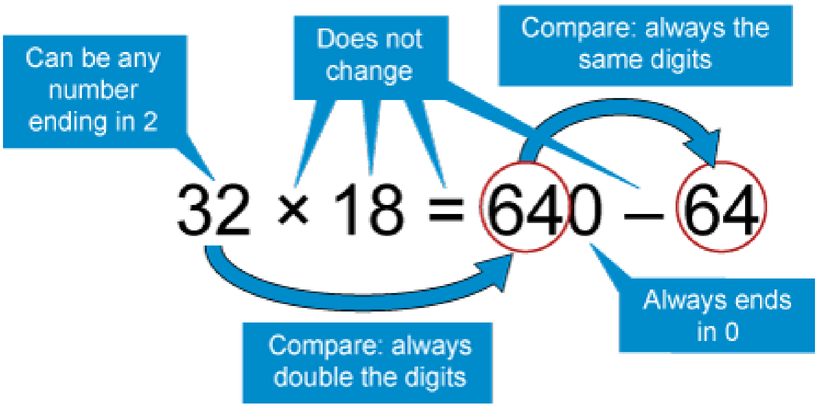
Save Reset

Discussion

It is difficult to explain general ideas without using examples. Sometimes you end up relying on words to describe positions in a given layout (first, second, left, right). Here you might say:

> The first expression you are going to write is a multiplication. Choose a whole number ending in two, and write multiplied by eighteen. The second expression is a subtraction. The first number is twenty times the whole number that appeared first in the multiplication (and so it always ends in four zero). Then write the subtraction sign. The second number is double that number (and always ends in four). If the two expressions are constructed in this way then they are equal, no matter what starting number.

Notice how hard it is to follow such an explanation without gestures or someone pointing to the layout.



An annotated number sentence

This image shows a number sentence with comments explaining what can be changed, and some features and relationships that have been noticed. Using arrows helps you point to features without naming them precisely.

Articulating your own thinking is essential for communicating to someone else what your general idea consists of and what general properties it has. It is also a first step to understanding why those properties work, which is the focus of the next task.

Task 1.7 Why are they equal?

Write an explanation of why you think these pairs of expressions are equal, in the general case. You could use words, numbers, diagrams or symbols.

Save Reset

Discussion

Here are some possible explanations.

Using words: one way of finding eighteen lots of something is to find twenty lots of something and then take away two lots of that thing.

In number sentences:

$$32 \times 18 = 32 \times (20 - 2) = 32 \times 20 - 32 \times 2 = 640 - 64$$

This reasoning works if I change 32 to any other number.

Word- or number-based expressions can be called mathematical sentences or number sentences, and they are the foundations of algebraic thinking. If a learner cannot read or write a number sentence then they cannot write an algebraic one.

More generally, using a symbol n to stand for the starting number:

$$n \times 18 = n \times (20 - 2) = n \times 20 - n \times 2 = 2n \times 10 - 2n$$

Throughout this task you have been asked to write down your thinking. This section has introduced you to some of the reasons for this:

- to support you in exploring generality
- to explain your methods to someone else
- to support you in understanding why your reasoning is valid.

The last part of this section summarises the overall approach of reasoning from examples.