The Open University

Faculty of Science, Technology, Engineering and Mathematics
MST326 Mathematical methods and fluid mechanics

MST326

Diagnostic Quiz

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Are you ready for MST326?

This diagnostic quiz is designed to help you decide if you are ready to study *Mathematical methods and fluid mechanics* (MST326). This document also contains some advice on preparatory work that you may find useful before starting MST326 (see below and page 5). The better prepared you are for MST326 the more time you will have to enjoy the module, and the greater your chance of success.

The topics which are included in this quiz are those that we expect you to be familiar with before you start the module. If you have previously studied *Mathematical methods and modelling* (MST210) or *Mathematical methods* (MST224) then you should be familiar with most of the topics covered in the quiz.

We suggest that you try this quiz first without looking at any books, and only look at a book when you are stuck. You will not necessarily remember everything; for instance, for the calculus questions you may need to look up a set of rules for differentiation or use the table of standard derivatives provided on page 2. This is perfectly all right, as such tables are provided in the Handbook for MST326. You need to check that you are able to use them though.

If you can complete the quiz with only the occasional need to look at other material, then you should be well prepared for MST326. If you find some topics that you remember having met before but need to do some more work on, then you should consider the suggestions for refreshing your knowledge on page 5.

Try the questions now, and then see the notes on page 5 of this document to see if you are ready for MST326.

Do contact your Student Support Team via StudentHome if you have any queries about MST326, or your readiness to study it.

Tables of standard derivatives and integrals

Function	Derivative
a	0
x^a	ax^{a-1}
e^{ax}	ae^{ax}
$\ln(ax)$	$\dfrac{1}{x}$
$\sin(ax)$	$a\cos(ax)$
$\cos(ax)$	$-a\sin(ax)$
$\tan(ax)$	$a\sec^2(ax)$
$\cot(ax)$	$-a\mathrm{cosec}^2(ax)$
$\sec(ax)$	$a\sec(ax)\tan(ax)$
$\mathrm{cosec}(ax)$	$-a\mathrm{cosec}(ax)\cot(ax)$
$\arcsin(ax)$	$\dfrac{a}{\sqrt{1-a^2x^2}}$
$\arccos(ax)$	$-\dfrac{a}{\sqrt{1-a^2x^2}}$
$\arctan(ax)$	$\dfrac{a}{1+a^2x^2}$
$\mathrm{arccot}(ax)$	$-\dfrac{a}{1+a^2x^2}$
$\mathrm{arcsec}(ax)$	$\dfrac{a}{\lvert ax\rvert\sqrt{a^2x^2-1}}$
$\mathrm{arccosec}(ax)$	$-\dfrac{a}{\lvert ax\rvert\sqrt{a^2x^2-1}}$

Function	Integral
a	ax
$x^a\ (a \neq -1)$	$\dfrac{x^{a+1}}{a+1}$
$\dfrac{1}{ax+b}$	$\dfrac{1}{a}\ln\lvert ax+b\rvert$
e^{ax}	$\dfrac{1}{a}e^{ax}$
$\ln(ax)$	$x(\ln(ax)-1)$
$\sin(ax)$	$-\dfrac{1}{a}\cos(ax)$
$\cos(ax)$	$\dfrac{1}{a}\sin(ax)$
$\tan(ax)$	$-\dfrac{1}{a}\ln\lvert\cos(ax)\rvert$
$\cot(ax)$	$\dfrac{1}{a}\ln\lvert\sin(ax)\rvert$
$\sec(ax)$	$\dfrac{1}{a}\ln\lvert\sec(ax)+\tan(ax)\rvert$
$\mathrm{cosec}(ax)$	$\dfrac{1}{a}\ln\lvert\mathrm{cosec}(ax)-\cot(ax)\rvert$
$\sec^2(ax)$	$\dfrac{1}{a}\tan(ax)$
$\mathrm{cosec}^2(ax)$	$-\dfrac{1}{a}\cot(ax)$
$\dfrac{1}{x^2+a^2}$	$\dfrac{1}{a}\arctan\left(\dfrac{x}{a}\right)$
$\dfrac{1}{(x-a)(x-b)}$	$\dfrac{1}{a-b}\ln\left\lvert\dfrac{a-x}{x-b}\right\rvert$
$\dfrac{1}{\sqrt{x^2+a^2}}$	$\ln(x+\sqrt{x^2+a^2})$ *or* $\mathrm{arcsinh}\left(\dfrac{x}{a}\right)$
$\dfrac{1}{\sqrt{x^2-a^2}}$	$\ln\lvert x+\sqrt{x^2-a^2}\rvert$ *or* $\mathrm{arccosh}\left(\dfrac{x}{a}\right)$
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\arcsin\left(\dfrac{x}{a}\right)$

Diagnostic quiz questions

1 Differentiation

Question 1

Differentiate the following functions with respect to x.

(a) $f(x) = 5\sin x + 3x^2 + 2\ln x$ (b) $g(x) = \cos 4x + e^{3x}$ (c) $h(x) = x^6 \tan 3x$

(d) $p(x) = \dfrac{5x^2 + x + 1}{2x + 3}$ (e) $q(x) = \sin(x^4 + 1)$

Question 2

Find the second derivative with respect to y of $f(y) = 5y^3 + 3\cos(2y)$.

2 Ordinary differential equations

Question 3

Find the general solution of each of the following differential equations:

(a) $\dfrac{dy}{dx} = e^{4x}$ (b) $\dfrac{dy}{dx} = \dfrac{y+1}{2x}$ $(x > 0)$

(c) $\dfrac{dy}{dx} = x^2 - \dfrac{2y}{x}$ $(x > 0)$ (d) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - 3y = 0$

Question 4

The perfect gas model for the atmospheric pressure p under isothermal conditions is given by

$$\frac{dp}{dz} = -\left(\frac{\rho_0 g}{p_0}\right) p,$$

where ρ_0, g and p_0 are constants. Find the particular solution that satisfies the condition $p(0) = p_0$.

3 Partial differentiation

Question 5

Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ when $f(x, y) = 2y^2 \cos x + e^{-(x^2+y^2)}$.

Question 6

Find all the second partial derivatives of $g(x, y) = 3x^2 \cos(x + y)$.

4 Vector algebra and calculus

Question 7

For the vectors $\mathbf{x} = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ and $\mathbf{y} = 5\mathbf{i} - 7\mathbf{j} + \mathbf{k}$, find

(a) $\mathbf{x} + \mathbf{y}$, (b) $\mathbf{x} \cdot \mathbf{y}$, (c) $\mathbf{x} \times \mathbf{y}$.

Question 8

(a) If $\mathbf{a} \cdot \mathbf{b} = 0$, what can be said about the vectors **a** and **b**?

(b) If $\mathbf{a} \times \mathbf{b} = 0$, what can be said about the vectors **a** and **b**?

Question 9

If $f(x, y, z) = 2xyz + xy^2 + xz^2$, find $\boldsymbol{\nabla} f$ (also known as $\mathbf{grad} f$) at the point $(1, 1, -2)$.

Question 10

If $\mathbf{F} = (xy + z^2)\mathbf{i} + x^2\mathbf{j} + (xz - 2)\mathbf{k}$, find $\boldsymbol{\nabla} \cdot \mathbf{F}$ (also known as divergence $\operatorname{div} \mathbf{F}$) and $\boldsymbol{\nabla} \times \mathbf{F}$ (also known as $\mathbf{curl}\,\mathbf{F}$).

Question 11

Consider the vector field $\mathbf{F} = \dfrac{1}{r}\mathbf{e}_r + r\cos\theta\mathbf{e}_\theta + (z^2 - 1)\mathbf{e}_z$ for $r > 0$, where r, θ, and z are cylindrical polar coordinates, find the divergence $\boldsymbol{\nabla} \cdot \mathbf{F}$.

Question 12

Calculate the line integral $\int_C \mathbf{F} \cdot d\mathbf{s}$ of the vector field $\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ along a path C with parametric equations $x = t$, $y = 1 + 2t$, $z = 4$ $(0 \leq t \leq 1)$.

5 Mechanics

Question 13

If $\mathbf{r} = 3\sin(2t)\mathbf{i} + 3\cos(2t)\mathbf{j} + 5t\mathbf{k}$ is the position vector at time t of a moving particle, find

(a) The velocity of the particle.

(b) The speed of the particle.

(c) The acceleration of the particle.

(d) A unit vector in the direction of the tangent to the trajectory of the particle at **r**.

Question 14

A box of mass $m = 10$ kilograms moves down a ramp which makes an angle θ with the horizontal, as shown in figure 1. Use the x-axis shown, which points down the slope with origin O at the top of the ramp and point A at the bottom of the ramp. Assume that the box can be modelled as a particle and that friction and air resistance can be neglected.

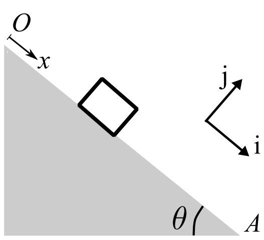

Figure 1 The box on the ramp.

(a) Sketch the force diagram of this system.

(b) If $\theta = \pi/4$ and the magnitude of the acceleration due to gravity is $g = 9.8$ m/s^2, find the acceleration of the box. Give the answer to two decimal places.

What can I do to prepare for MST326?

Now that you have finished the quiz, how did you get on? The information below should help you to decide what to do next.

Most of this quiz was unfamiliar to me. I don't think I have met any of the topics before. →	You should consider taking MST210 or MST224 before MST326. Contact your Student Support Team to discuss this.
Some of these topics are familiar to me, but not all of them. →	If you have met less than 75% of these topics then you should consider taking MST210 or MST224 before MST326. Contact your Student Support Team to discuss this.
I have met all these topics before, but I am a bit rusty on some, and had to look up how to do a few of them. →	You will benefit from doing some revision before starting MST326. A good starting point is the MST326 *Revision Booklet*. See below for possible other sources of help.
I can do all these questions. →	You are probably well prepared for MST326, but do brush up on any topics you need to further enhance your confidence by going over the MST326 *Revision Booklet*. You might also wish to get ahead with your MST326 studies. Early content is available to registered students under DISCOVER...*Discover your module* at learn2.open.ac.uk/site/s-maths.

What resources are there to help me prepare for MST326?

If you need to do some background preparation before starting MST326, we suggest you concentrate your efforts on the following topics.

- Differentiation (both for functions of one variable and functions of several variables)
- Simple ordinary differential equations
- Vector algebra and calculus (particularly the gradient, divergence and curl of a function of several variables)
- Newtonian mechanics

There are many undergraduate textbooks covering these topics. The following are available to registered students online through the OU Library.

- Lipschutz, S. and Lipson, M., "Schaum's Outline of Linear Algebra", Fourth Edition, McGraw-Hill, ISBN: 9780071543521
- Bear, H.S., "Understanding calculus", IEEE-Wiley
- Ayres, F. and Mendelson, E., "Schaum's Outline of Calculus", Fifth edition, McGraw-Hill, ISBN: 9780071508612
- Wrede, R. and Spiegel, M., "Schaum's Outline of Advanced Calculus", Third edition, McGraw-Hill, ISBN: 9780071623667
- Chapters 1 and 2 of McCall, M.W., "Classical Mechanics", Second edition, Wiley, ISBN: 9780470715727

If you have studied MST210, its predecessor MST209, or MST224, then you could use parts of these to revise for MST326.

The mathcentre web-site (`www.mathcentre.ac.uk`) includes several teach-yourself books, summary sheets, revision booklets, online exercises and video tutorials on a range of mathematical skills.

If you have any queries contact your Student Support Team via StudentHome.

Diagnostic quiz solutions

Solution to question 1

(a) $f'(x) = 5\cos x + 6x + \dfrac{2}{x}$

(b) $g'(x) = -4\sin 4x + 3e^{3x}$

(c) $h'(x) = 6x^5 \tan 3x + 3x^6 \sec^2 3x$

(d) $$\begin{aligned} p(x) &= \frac{(10x+1)(2x+3) - (5x^2+x+1)(2)}{(2x+3)^2} \\ &= \frac{20x^2 + 30x + 2x + 3 - 10x^2 - 2x - 2}{(2x+3)^2} \\ &= \frac{10x^2 + 30x + 1}{(2x+3)^2} \end{aligned}$$

(e) $q(x) = 4x^3 \cos(x^4 + 1)$

Solution to question 2

$f'(y) = 15y^2 - 6\sin(2y)$, so $f''(y) = 30y - 12\cos(2y)$.

Solution to question 3

(a) By direct integration: $\int dy = \int e^{4x} dx$ and so $y(x) = \frac{1}{4}e^{4x} + C$, where C is an arbitrary constant.

(b) By using the method of separation of variables for first-order differential equations $\int \dfrac{dy}{y+1} = \int \dfrac{dx}{2x}$ from which we find $\ln(y+1) = \frac{1}{2}\ln x + C$, where C is an arbitrary constant. Writing $C = \ln A$, where A is another constant, and rearranging, we find $y + 1 = Ax^{1/2}$. Therefore $y(x) = Ax^{1/2} - 1$ with $x > 0$.

(c) This equation can be solved by using the integrating factor method. The integrating factor is $p(x) = \exp \int \frac{2}{x} dx = x^2$ and we can write $\frac{d}{dx}(x^2 y) = x^4$. We solve this equation by direct integration to find the general solution $y(x) = \frac{x^3}{5} + \frac{C}{x^2}$ where C is an arbitrary constant.

(d) This is a homogeneous linear second-order differential equation with constant coefficients. The auxiliary equation is $\lambda^2 + 2\lambda - 3 = 0$ which has roots $\lambda_1 = 1$ and $\lambda_2 = -3$. Therefore, the general solution is $y(x) = Ae^x + Be^{-3x}$.

Solution to question 4

We use the separation of variables method to write:

$$\int \frac{dp}{p} = -\int \left(\frac{\rho_0 g}{p_0}\right) dz,$$

so that we have $\ln p = \left(-\dfrac{\rho_0 g}{p_0}\right) z + C$ where C is an arbitrary constant. By writing $C = \ln A$, where A is another constant, and rearranging we arrive at the general solution $p(z) = A\exp\left(-\dfrac{\rho_0 g}{p_0} z\right)$. By imposing the condition $p(0) = p_0$, we find $A = p_0$ and so the particular solution is $p(z) = p_0 \exp\left(-\dfrac{\rho_0 g}{p_0} z\right)$.

Solution to question 5

$$\frac{\partial f}{\partial x} = -2y^2 \sin x - 2xe^{-(x^2+y^2)}.$$
$$\frac{\partial f}{\partial y} = 4y \cos x - 2ye^{-(x^2+y^2)}.$$

Solution to question 6

The first partial derivatives are

$$\frac{\partial g}{\partial x} = 6x\cos(x+y) - 3x^2 \sin(x+y)$$
$$\frac{\partial g}{\partial y} = -3x^2 \sin(x+y).$$

The second partial derivatives are

$$\frac{\partial^2 g}{\partial x^2} = 6\cos(x+y) - 12x\sin(x+y) - 3x^2\cos(x+y)$$
$$\frac{\partial^2 g}{\partial x \partial y} = -6x\sin(x+y) - 3x^2\cos(x+y)$$
$$\frac{\partial^2 g}{\partial y^2} = -3x^2\cos(x+y)$$
$$\frac{\partial^2 g}{\partial y \partial x} = -6x\sin(x+y) - 3x^2\cos(x+y).$$

Solution to question 7

(a) $\mathbf{x} + \mathbf{y} = 8\mathbf{i} - 5\mathbf{j} + 5\mathbf{k}$

(b) $\mathbf{x} \cdot \mathbf{y} = 15 - 14 + 4 = 5$

(c) $\mathbf{x} \times \mathbf{y} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 4 \\ 5 & -7 & 1 \end{vmatrix} = (2+28)\mathbf{i} + (20-3)\mathbf{j} + (-21-10)\mathbf{k} = 30\mathbf{i} + 17\mathbf{j} - 31\mathbf{k}$

Solution to question 8

(a) If $\mathbf{a} \cdot \mathbf{b} = 0$, then either $\mathbf{a} = 0$ or $\mathbf{b} = 0$, or $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.

(b) If $\mathbf{a} \times \mathbf{b} = 0$ then either $\mathbf{a} = 0$ or $\mathbf{b} = 0$, or $\mathbf{a}$ and $\mathbf{b}$ are parallel (in the same or opposite directions).

Solution to question 9

$$\begin{aligned}\nabla f &= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \\ &= (2yz + y^2 + z^2)\mathbf{i} + (2xz + 2xy)\mathbf{j} + (2xy + 2zx)\mathbf{k}\end{aligned}$$

So at $(1, 1, -2)$, $\quad \nabla f(1, 1, -2) = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$

Solution to question 10

If $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k} = (xy + z^2)\mathbf{i} + x^2\mathbf{j} + (xz - 2)\mathbf{k}$ then we have:

$$\nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = y + x$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix} = (0-0)\mathbf{i} + (2z - z)\mathbf{j} + (2x - x)\mathbf{k} = z\mathbf{j} + x\mathbf{k}$$

Solution to question 11

If $\mathbf{F} = F_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta + F_z\mathbf{e}_z = \frac{1}{r}\mathbf{e}_r + r\cos\theta\mathbf{e}_\theta + (z^2 - 1)\mathbf{e}_z$ in cylindrical polar coordinates, then:

$$\nabla \cdot \mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z} = 0 - \sin\theta + 2z = 2z - \sin\theta.$$

Solution to question 12

Differentiating the parametric equations gives

$\frac{dx}{dt} = 1, \qquad \frac{dy}{dt} = 2, \qquad \frac{dz}{dt} = 0.$

Expressing now the components of $\mathbf{F}$ in terms of t, we get

$F_1 = yz = 4(1 + 2t), \qquad F_2 = xz = 4t, \qquad F_3 = xy = t(1 + 2t).$

Hence we have $\mathbf{F} \cdot \frac{d\mathbf{s}}{dt} = 4(1 + 2t) + 8t = 4 + 16t$. The required line integral is therefore:

$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_C \mathbf{F} \cdot \frac{d\mathbf{s}}{dt} dt = \int_0^1 (4 + 16t)dt = 12.$

Solution to question 13

(a) The velocity of the particle is $\mathbf{v} = \frac{d\mathbf{r}}{dt} = 6\cos(2t)\mathbf{i} - 6\sin(2t)\mathbf{j} + 5\mathbf{k}$.

(b) The speed of the particle is $v = |\mathbf{v}| = \sqrt{36\cos^2(2t) + 36\sin^2(2t) + 25} = \sqrt{61}$.

(c) The acceleration of the particle is $\mathbf{a} = \frac{d\mathbf{v}}{dt} = -12\sin(2t)\mathbf{i} - 12\cos(2t)\mathbf{j}$.

(d) The derivative $d\mathbf{r}/dt$ at a point on the curve is a tangent vector to the curve at that point. A unit vector in the direction of this tangent vector is $\mathbf{e}_t = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{61}}(6\cos(2t)\mathbf{i} - 6\sin(2t)\mathbf{j} + 5\mathbf{k})$.

Solution to question 14

(a) Since we are neglecting friction and air resistance, the only forces on the box are the weight **W** and the normal reaction **N**. Therefore, the force diagram is as shown in figure 2.

(b) Newton's second law for this system gives $m\mathbf{a} = \mathbf{W} + \mathbf{N}$. From the force diagram, we have $\mathbf{N} = |N|\mathbf{j}$. The weight can be resolved into components as $\mathbf{W} = |\mathbf{W}|\sin\theta\mathbf{i} - |\mathbf{W}|\cos\theta\mathbf{j}$. The motion is along the slope, that is $\mathbf{a} = a\mathbf{i}$. Therefore, taking the weight of the box to be $|\mathbf{W}| = mg$, in the $\mathbf{i}$ direction we have $ma = mg\sin\theta$, which gives $a = g\sin\theta$. Putting the given numerical values, we find $a = 6.93\ \text{m/s}^2$.

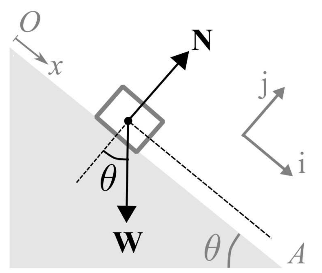

Figure 2 Force diagram.