M208
Diagnostic quiz

Am I ready to start M208?

The diagnostic quiz below is designed to help you to answer this question. This document also contains some advice on preparatory work that you may find useful before starting M208 (see below and page 6).

The topics which are included in the first three sections of the quiz are those that we expect you to be familiar with before you start the module. Ideally, you would also be familiar with the topics covered in the remaining sections although these topics are all revised in M208. If you have previously studied MST124 and MST125 then you should be familiar with all the topics covered in the quiz.

We suggest that you try this quiz first without looking at any books or using a calculator, and only look at a book when you are stuck. You will not necessarily remember everything; for instance, for the calculus questions you may need to look up a set of rules for differentiation or integration or a table of standard derivatives or integrals. This is perfectly all right, as such tables are provided in the Handbook for M208. You need to check that you are able to use them though.

If you can complete the quiz with only the occasional need to look at other material, then you should be well prepared for M208. If you find some topics that you remember having met before but need to do some more work on, then you should consider the suggestions for refreshing your knowledge on page 6.

M208 is a very rewarding course, covering a wide range of pure mathematics. The better prepared you are for it the more time you will have to enjoy the mathematics, and the greater your chance of success.

Try the questions now, and then see the notes on page 6 of this document to see if you are ready for M208. (The answers to the questions begin on page 8.)

Do contact your Student Support Team via StudentHome if you have any queries about M208, or your readiness to study it.

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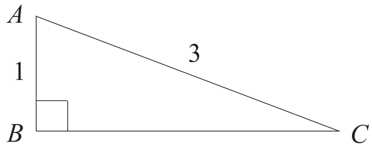
Diagnostic quiz questions

Introductory questions:

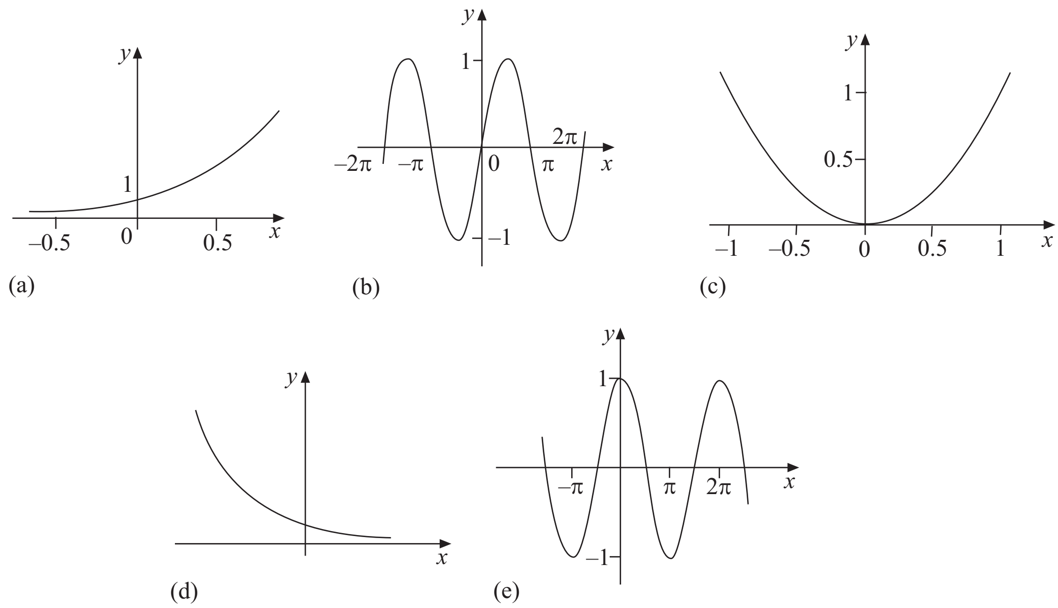
1. In a right-angled triangle ABC with right-angle at B, AB has length 1 and AC has length 3. What is the length of BC?



2. What is the equation of the straight line through the points $(2, 1)$ and $(4, 5)$? What is the gradient (or slope) of this line?

3. Five graphs are given in parts (a)–(e) of the figure below. Each graph is that of one of the functions (i)–(v). Match each graph with the appropriate function.

Functions: (i) $f(x) = e^{-2x}$; (ii) $f(x) = e^{2x}$; (iii) $f(x) = \sin x$; (iv) $f(x) = \cos x$; (v) $f(x) = x^2$.



4. What are the exact values of

$$\cos\left(\tfrac{1}{2}\pi\right), \quad \sin\left(\tfrac{1}{4}\pi\right), \quad \tan\left(\tfrac{1}{4}\pi\right), \quad \cos\left(\tfrac{1}{3}\pi\right),$$

$$\sin \tfrac{1}{3}\pi, \quad \cos\left(\tfrac{1}{6}\pi\right), \quad \sin\left(\tfrac{1}{6}\pi\right)?$$

5. What are the exact values of

$$\cos\left(\tfrac{3}{4}\pi\right), \quad \sin\left(\tfrac{7}{4}\pi\right), \quad \tan\left(\tfrac{5}{4}\pi\right)?$$

6. If $\sin\theta = \frac{3}{5}$, what are the possible values of $\cos\theta$?

Basic algebra:

1. Solve each of the following equations for x.

(a) $2x + 7 = 13$ (b) $3(x + 3) = 7(x - 1)$

(c) $\dfrac{3}{1 - x} = \dfrac{2}{2 + x}$ (d) $3x^2 - x = 0$

(e) $2x^2 - 5x - 3 = 0$ (f) $2x^2 + 7x + 4 = 0$

2. Solve the following simultaneous equations for x and y:

$$\begin{aligned} 2x - 3y &= 4, \\ x + 2y &= 9. \end{aligned}$$

3. (a) Make u the subject of the equation

$$t^2 = \frac{2(s - ut)}{a}.$$

(b) Make x the subject of the equation

$$\sqrt{\frac{x-2}{x+3}} = t.$$

4. Simplify each of the following expressions.

(a) $x^3 x^4$

(b) x^2/x^5

(c) $\left(x^3\right)^2$

(d) $4^{1/2}$

(e) $\left(e^{-2x} \times e^{3x}\right)^2$

5. Simplify each of the following expressions.

(a) $t(2t + 3) - 2t(1 - 3t)$

(b) $\dfrac{21a^5b^7}{49a^3b^{10}}$

(c) $\dfrac{12p^2 + 20p + 8}{6p^2 + 7p + 2}$

6. Express each of the following as the product of two linear factors:

(a) $16x^2 - 25$ (b) $3y^2 - 5y + 2$ (c) $6t^2 - t - 15$

(d) $100a^2 - 9b^2$

7. Use your answer to Question 6(d) to find the value of 53×47 without using a calculator (or doing "long multiplication" by hand!)

8. Express each of the following in completed-square form (that is, as $a(x + r)^2 + s$, where a, r and s are real numbers):

(a) $x^2 + 3x + 5$ (b) $2x^2 - 8x + 7$ (c) $-3x^2 - 8x + 1$

9. Each of the expressions in Question 8 reaches either a maximum or a minimum value for some real value of x. Using your completed-square expressions, find this value, and the corresponding value of x, in each case.

10. Find the solution set of the inequality $\dfrac{2 + x}{5x - 3} \geq 8$, giving your answer in interval notation.

11. Simplify the following:

(a) $\dfrac{3^{n+2}(n+1)!}{3^{n+1}(n+2)!}$ (b) $\dfrac{(n+1)^2(2n)!}{n^2(2(n+1))!}$

12. Rationalise the denominators of the following:

(a) $\dfrac{3\sqrt{2}+2\sqrt{3}}{\sqrt{3}+\sqrt{2}}$ (b) $\dfrac{4-3\sqrt{5}}{2\sqrt{5}+1}$

13. If $f(x,y) = x^2 + xy - 2y^2$, write down (and simplify, if possible) the following expressions:

(a) $f(x,x)$ (b) $f(y,x)$ (c) $f(y,z)$ (d) $f(-2x,3y)$

14. Find the sum of the infinite series $3 - 3\left(\frac{5}{7}\right) + 3\left(\frac{5}{7}\right)^2 - 3\left(\frac{5}{7}\right)^3 + \ldots$

Calculus:

1. Differentiate the following functions with respect to x.

(a) $f(x) = x^4 + 5x^3 - x^2 + 2x - 1$

(b) $f(x) = \sin x$

(c) $f(x) = 3e^{2x}$

2. Differentiate the following functions with respect to x.

(a) $f(x) = (x^3+3)\cos x$ (b) $f(x) = e^x(x^2+5x-3)$

(c) $f(x) = \dfrac{5}{x^2-1}$ (d) $f(x) = \dfrac{x^2-x+1}{2x+3}$

3. (a) Find $f'(x)$, where $f(x) = e^{3\sin x}$.

(b) Find $f'(x)$, where $f(x) = \cos\left(3x^2+2x-6\right)$.

4. Evaluate each of the following integrals.

(a) $\displaystyle\int (2x^3+5)\,dx$ (b) $\displaystyle\int_0^{\pi/2} \cos(5t)\,dt$

5. (a) Use integration by parts to evaluate $\displaystyle\int x\sin x\,dx$.

(b) Use integration by substitution to evaluate $\displaystyle\int x^2 e^{4x^3}\,dx$.

Complex numbers:

1. Find, in the form $az^2 + bz + c = 0$ where a, b and c are integers, a quadratic equation whose solutions are $\frac{1}{5} \pm 2i$.

2. Let $z = 3 - i$ and $w = -2 + 2i$. Find:

(a) $z + w$ (b) $\dfrac{z}{w}$ (c) $|w|$ (d) $\overline{z}$

(e) $\text{Arg}(w)$ [that is, the principal value of the argument]

3. Use your answers to Question 2(c) and (e) to express $w = -2 + 2i$ in polar form, and hence find w^6, converting your answer to Cartesian form.

Vectors:

1. Consider the vectors $\mathbf{a}$ and $\mathbf{b}$ in the plane with component forms $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$, $\mathbf{b} = -6\mathbf{i} + 4\mathbf{j}$.

(a) Find the following:

(i) $\mathbf{a} - \mathbf{b}$ (ii) $3\mathbf{a} + \mathbf{b}$

(iii) The magnitude of $\mathbf{b}$, and the sine and cosine of the angle it makes with the positive x-direction.

(b) Show that $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.

2. The vector $\mathbf{c}$ is of magnitude 8 and makes an angle of $\frac{7\pi}{4}$ with the positive x-direction. Find $\mathbf{c}$ in component form.

Matrices:

1. Let $\mathbf{A} = \begin{pmatrix} 6 & 0 \\ -2 & 3 \\ 0 & 5 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 4 & 2 \\ 6 & 3 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$.

(a) For each of the following, either evaluate it, or explain why evaluation is not possible:

(i) $\mathbf{AB}$ (ii) $\mathbf{BA}$ (iii) $2\mathbf{A} + \mathbf{B}$ (iv) $\mathbf{B} - \mathbf{C}$

(v) $\mathbf{A}^2$ (vi) $\mathbf{B}^2$ (vii) $\mathbf{C}^{-1}$ (viii) $\mathbf{B}^{-1}$

(b) Use an answer from (a) to solve the following pair of simultaneous equations:

$$\begin{aligned} 3x + 2y &= 3, \\ x + 4y &= 4. \end{aligned}$$

(c) Find the eigenvalues of $\mathbf{C}$, and find corresponding eigenvectors with integer components.

Methods of proof:

1. Consider the following statement about integers n:

If n^2 is divisible by 12, then n is even.

(a) Give a proof by contraposition that the statement is true.

(b) Write down the converse of the statement. Is the converse true or false? Give a proof or counterexample, as appropriate.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be the function with rule $f(x) = 5x - 3$. Show that f is one-to-one.

3. Prove that the congruence $x^3 \equiv 2 \pmod{7}$ has no solutions. What method of proof have you used?

4. (a) Give a proof by contradiction to show that the sum of a rational number and an irrational number must be irrational.

(b) Give a proof by construction to show that the sum of two irrational numbers may be rational.

5. Use mathematical induction to prove that the following statements about natural numbers n are true for the given values of n:

(a) $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2, \quad \text{for } n = 1, 2 \ldots$

(b) $2^n < n!, \quad \text{for } n \geq 4$

What can I do to prepare for M208?

Now that you have finished the quiz, how did you get on? The information below should help you to decide what, if anything, you should do next.

Most of this quiz was unfamiliar to me. I don't think I have met any of the topics before. →	Perhaps you should study another module such as MU123 or MST124. Contact your Student Support Team to discuss this.
Some of these topics are familiar to me, but not all of them. →	Consider taking MST124 and MST125 first. (Or MST125 if you have already taken MST124.) Contact your Student Support Team.
I have met all these topics before, but I am a bit rusty on some, and had to look up how to do a few of them. →	You will benefit from doing some revision before starting M208 – see below for some possible sources of help.
I can do all of these questions. →	You are probably well prepared for M208, but do brush up on any topics you need to further enhance your confidence.

If you have any queries contact your Student Support Team via StudentHome.

What resources are there to help me revise for M208?

If you have studied MST124 and MST125, or their predecessors MST121 and MS221 before, then you could use parts of these to revise for M208.

If you need to brush up some of the more basic topics like algebra, trigonometry and calculus, then you may find some of the textbooks designed for A-level pure mathematics useful. Alternatively, there are some suitable books in the Teach Yourself series published by Hodder and Stoughton.

There are also two books called *Countdown to Mathematics*, Volumes 1 and 2, by Lynne Graham and David Sargent, which have plenty of examples for practice. They are published by Addison Wesley, and their ISBN numbers are 201 13730 5 for Volume 1 and 201 13731 3 for Volume 2.

It is also important to revise topics such as mathematical proof and matrices. There are many websites that offer revision materials in mathematics that you may find useful for revising these and many other topics. However the sites listed below are not Open University websites, so we cannot guarantee that they will continue to be available.

The Maths Support Centre

http://www.mathcentre.ac.uk/
This site has teach-yourself booklets, summary sheets, revision booklets, online exercises and video tutorials on a wide range of topics to help you to develop the mathematical skills needed for M208.

Calculus on the web

http://cow.math.temple.edu/
This is an interactive site with examples on algebra, calculus, linear algebra, abstract algebra and number theory.

Other resources

You can also look for other resources on the web. If you go to the Open University Library

https://learn2.open.ac.uk/mod/subpage/view.php?id=634333

you can access the following resources:

Ebooks

Thinking mathematically 2nd edition. Mason, J., Burton, L. and Stacey, K. (2010)

The tiger that isnt Blastland, M. and Dilnot, A. (2007)

Why do buses come in threes? Eastaway,R. and Wyndham, J. (1998)
(4 hour 'loan'. Just use the eBook full text link to open.)

Prime obsession. Derbyshire, J. (2003)

Journals/magazines

Mathematical Intelligencer
Informal articles about mathematics, mathematicians and the history and culture of mathematics.

New Scientist
This popular science magazine also has articles on mathematics.

Mathematical Gazette
Articles on the learning and teaching of mathematics. (Access from 1894. Last five years not available.)

BSHM Bulletin Journal of the British Society for the History of Mathematics
Contains very readable articles on the history of mathematics from all periods and civilizations, and all aspects of mathematics.

Diagnostic quiz answers

Introductory questions:

1. By Pythagoras' Theorem,

$$AC^2 = AB^2 + BC^2$$
$$9 = 1 + BC^2,$$

so

$$BC^2 = 9 - 1 = 8$$
$$BC = \sqrt{8} = 2\sqrt{2}.$$

[Ref: MST124 Unit 4.]

2. The general equation of a straight line is $y = mx + c$. Substituting the coordinates of P and Q into this equation gives

$$1 = 2m + c,$$
$$5 = 4m + c.$$

Subtracting the first equation from the second gives

$$4 = 2m,$$

that is, $m = 2$.

Using the first equation gives

$$1 = 4 + c,$$

that is, $c = -3$.

So the equation of the line is $y = 2x - 3$. The gradient of the line is 2, as $m = 2$. (There are other ways of doing this question.)

[Ref: MST124 Unit 2.]

3. The matching is as follows:

(a) (ii); (b) (iii); (c) (v); (d) (i); (e) (iv).

(Graphs like these are revised in the first unit of M208.)

[Ref: MST124 Units 2, 3 and 4.]

4. $\cos\left(\frac{1}{2}\pi\right) = 0, \quad \sin\left(\frac{1}{4}\pi\right) = \frac{1}{\sqrt{2}}, \quad \tan\left(\frac{1}{4}\pi\right) = 1,$

$\cos\left(\frac{1}{3}\pi\right) = \frac{1}{2}, \quad \sin\left(\frac{1}{3}\pi\right) = \frac{\sqrt{3}}{2}, \quad \cos\left(\frac{1}{6}\pi\right) = \frac{\sqrt{3}}{2},$

$\sin\left(\frac{1}{6}\pi\right) = \frac{1}{2}.$

[Ref: MST124 Unit 4.]

5. $\cos\left(\frac{3}{4}\pi\right) = -\cos\left(\frac{1}{4}\pi\right) = -\dfrac{1}{\sqrt{2}}$,

$$\sin\left(\tfrac{7}{4}\pi\right) = \sin\left(2\pi - \tfrac{1}{4}\pi\right) = \sin\left(-\tfrac{1}{4}\pi\right) = -\sin\left(\tfrac{1}{4}\pi\right) = -\frac{1}{\sqrt{2}},$$

$\tan\left(\frac{5}{4}\pi\right) = \tan\left(\frac{1}{4}\pi\right) = 1.$

[Ref: MST124 Unit 4.]

6. We have the identity $\sin^2\theta + \cos^2\theta = 1$.

So, if $\sin\theta = \frac{3}{5}$, then

$$\begin{aligned}\left(\tfrac{3}{5}\right)^2 + \cos^2\theta &= 1\\ \cos^2\theta &= 1 - \left(\tfrac{3}{5}\right)^2\\ &= 1 - \frac{9}{25} = \frac{16}{25},\end{aligned}$$

so $\cos\theta = \pm\frac{4}{5}$.

[Ref: MST124 Unit 4.]

Basic algebra:

1. We solve these equations by rearranging them into equivalent simpler forms.

(a)
$$\begin{aligned}2x + 7 &= 13\\ 2x &= 6\\ x &= 3\end{aligned}$$

(b)
$$\begin{aligned}3(x+3) &= 7(x-1)\\ 3x + 9 &= 7x - 7\\ 3x - 7x &= -7 - 9\\ -4x &= -16\\ x &= 4\end{aligned}$$

(c)
$$\begin{aligned}\frac{3}{1-x} &= \frac{2}{2+x}\\ 3(2+x) &= 2(1-x)\\ 6 + 3x &= 2 - 2x\\ 5x &= -4\\ x &= -\tfrac{4}{5}\end{aligned}$$

(d)
$$\begin{aligned}3x^2 - x &= 0\\ x(3x-1) &= 0.\end{aligned}$$

Hence $x = 0$ or $3x - 1 = 0$.
So $x = 0$ or $x = \frac{1}{3}$,
so the solutions are 0 or $\frac{1}{3}$.

(e)
$$\begin{aligned}2x^2 - 5x - 3 &= 0\\ (2x+1)(x-3) &= 0.\end{aligned}$$

Hence $2x + 1 = 0$ or $x - 3 = 0$.
So $x = -\frac{1}{2}$ or $x = 3$,
so the solutions are $-\frac{1}{2}$ and 3.

(f) Using the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, where $a = 2$, $b = 7$ and $c = 4$ gives

$$\begin{aligned} x &= \frac{-7 \pm \sqrt{7^2 - 4.2.4}}{4} \\ &= \frac{-7 \pm \sqrt{49 - 32}}{4} \\ &= \frac{-7 \pm \sqrt{17}}{4}, \end{aligned}$$

so the solutions are $\dfrac{-7 + \sqrt{17}}{4}$ and $\dfrac{-7 - \sqrt{17}}{4}$.

[Ref: MST124 Units 1 and 2.]

2. $2x - 3y = 4,$
$x + 2y = 9.$

Multiplying the second equation by 2 gives

$$\begin{aligned} 2x - 3y &= 4 \\ 2x + 4y &= 18. \end{aligned}$$

Subtracting the first equation from the second gives

$$7y = 14,$$

hence $y = 2$.
Substituting for y into the second original equation gives

$$x + 4 = 9,$$

hence $x = 5$.

So the solution is $x = 5$, $y = 2$.

[Ref: MST124 Unit 2.]

3. (a)
$$\begin{aligned} t^2 &= \frac{2(s - ut)}{a} \\ \Leftrightarrow at^2 &= 2(s - ut) = 2s - 2ut \\ \Leftrightarrow 2ut &= 2s - at^2 \\ \Leftrightarrow \quad u &= \frac{2s - at^2}{2t} \end{aligned}$$

(b) $\sqrt{\dfrac{x-2}{x+3}} = t$

$\Rightarrow \dfrac{x-2}{x+3} = t^2,$

so $x - 2 = t^2(x + 3) = t^2 x + 3t^2$.

Rearranging this equation gives

$$\begin{aligned} x(1 - t^2) &= 2 + 3t^2 \\ x &= \frac{2 + 3t^2}{1 - t^2}. \end{aligned}$$

[Ref: MST124 Unit 2.]

4. (a) $x^3x^4 = x^{3+4} = x^7$

(b) $x^2/x^5 = x^{2-5} = x^{-3}$

(c) $(x^3)^2 = x^{3\times 2} = x^6$

(d) $4^{1/2} = \sqrt{4} = 2$

(e) $\left(e^{-2x} \times e^{3x}\right)^2 = \left(e^{-2x+3x}\right)^2 = (e^x)^2 = e^{2x}$

[Ref: MST124 Units 1 and 3.]

5. (a) $t(2t+3) - 2t(1-3t)$
$= 2t^2 + 3t - 2t + 6t^2$
$= 8t^2 + t$

(b)
$$\frac{21a^5b^7}{49a^3b^{10}} = \frac{21}{49} \times \frac{a^5}{a^3} \times \frac{b^7}{b^{10}} = \frac{3}{7} \times \frac{a^2}{b^3} = \frac{3a^2}{7b^3}$$

(c)
$$\frac{12p^2+20p+8}{6p^2+7p+2} = \frac{4(3p^2+5p+2)}{(3p+2)(2p+1)} = \frac{4(3p+2)(p+1)}{(3p+2)(2p+1)} = \frac{4(p+1)}{(2p+1)}$$

[Ref: MST124 Units 1 and 2.]

6. (a) $16x^2 - 25 = (4x)^2 - 5^2 = (4x+5)(4x-5)$

(b) $3y^2 - 5y + 2 = (3y-2)(y-1)$

(c) $6t^2 - t - 15 = (3t-5)(2t+3)$

(d) $100a^2 - 9b^2 = (10a)^2 - (3b)^2 = (10a+3b)(10a-3b)$

[Ref: MST124 Units 1 and 2.]

7. Taking $a = 5$, $b = 1$ in Q6(d) above gives $10a + 3b = 53$ and $10a - 3b = 47$, so

$$53 \times 47 = 100(5)^2 - 9(1)^2 = 2500 - 9 = 2491.$$

8. (a) $$x^2+3x+5=\left[\left(x+\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2\right]+5$$
$$=\left(x+\frac{3}{2}\right)^2-\frac{9}{4}+\frac{20}{4}=\left(x+\frac{3}{2}\right)^2+\frac{11}{4}$$

(b) $$2x^2-8x+7=2(x^2-4x)+7=2[(x-2)^2-(-2)^2]+7=2(x-2)^2-8+7$$
$$=2(x-2)^2-1$$

(c) $$-3x^2-8x+1=-3\left(x^2+\frac{8}{3}x\right)+1=-3\left[\left(x+\frac{4}{3}\right)^2-\left(\frac{4}{3}\right)^2\right]+1$$
$$=-3\left(x+\frac{4}{3}\right)^2+\frac{16}{3}+1=-3\left(x+\frac{4}{3}\right)^2+\frac{19}{3}$$

[Ref: MST124 Unit 2.]

9. $(x+r)^2\geq 0$ for all real numbers x, and is zero when $x+r=0$, that is, $x=-r$. So the expression $a(x+r)^2$ too is zero when $x=-r$, and takes the same sign as a for all other values of x. Hence the expression $a(x+r)^2+s$ takes its extreme value s when $x=-r$; if $a>0$, this extreme value is a minimum, and if $a<0$, it is a maximum.

Hence the expressions in Question 8 reach these extreme values:

(a) minimum $\frac{11}{4}$ when $x=-\frac{3}{2}$

(b) minimum -1 when $x=2$

(c) maximum $\frac{19}{3}$ when $x=-\frac{4}{3}$

[Ref: MST124 Unit 2.]

10. $$\frac{2+x}{5x-3}\geq 8 \iff \frac{2+x}{5x-3}-8\geq 0 \iff \frac{(2+x)-8(5x-3)}{5x-3}$$
$$\geq 0 \iff \frac{2+x-40x+24}{5x-3}\geq 0 \iff \frac{-39x+26}{5x-3}\geq 0 \iff \frac{13(-3x+2)}{5x-3}\geq 0.$$

We now draw up a table of signs for the expression on the left; the numerator is zero when $x=\frac{2}{3}$, and the denominator is zero when $x=\frac{3}{5}$.

x	$(-\infty,3/5)$	$3/5$	$(3/5,2/3)$	$2/3$	$(2/3,\infty)$
$13(-3x+2)$	$+$	$+$	$+$	0	$-$
$5x-3$	$-$	0	$+$	$+$	$+$
Whole expression	$-$	$*$	$+$	0	$-$

So the solution set is the interval $\left(\frac{3}{5},\frac{2}{3}\right]$. Note that the right-hand endpoint is included, since equality holds there; the left-hand endpoint is excluded, since the expression is undefined there.

[Ref: MST124 Unit 3.]

11. (a) $\dfrac{3^{n+2}(n+1)!}{3^{n+1}(n+2)!} = \dfrac{3 \times 3^{n+1}(n+1)!}{3^{n+1}(n+2)(n+1)!} = \dfrac{3}{n+2}$

(b) $\dfrac{(n+1)^2(2n)!}{n^2(2(n+1))!} = \dfrac{(n+1)^2(2n)!}{n^2(2n+2)!} = \dfrac{(n+1)^2(2n)!}{n^2(2n+2)(2n+1)(2n)!}$

$$= \frac{(n+1)^2}{n^2(2n+2)(2n+1)} = \frac{(n+1)^2}{n^2.2(n+1)(2n+1)} = \frac{n+1}{2n^2(2n+1)}$$

[Ref: MST124 Unit 1 (powers), MST124 Unit 10, MST125 Unit 12 (factorials).]

12. (a) $\dfrac{3\sqrt{2}+2\sqrt{3}}{\sqrt{3}+\sqrt{2}} = \left(\dfrac{3\sqrt{2}+2\sqrt{3}}{\sqrt{3}+\sqrt{2}}\right)\left(\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}\right) = \dfrac{3\sqrt{6}-6+6-2\sqrt{6}}{3-2} = \sqrt{6}$

(b) $\dfrac{4-3\sqrt{5}}{2\sqrt{5}+1} = \left(\dfrac{4-3\sqrt{5}}{2\sqrt{5}+1}\right)\left(\dfrac{2\sqrt{5}-1}{2\sqrt{5}-1}\right) = \dfrac{8\sqrt{5}-4-30+3\sqrt{5}}{20-1} = \dfrac{11\sqrt{5}-34}{19}$

[Ref: MST124 Unit 1.]

13. (a) $f(x,x) = x^2 + x^2 - 2x^2 = 0$

(b) $f(y,x) = y^2 + yx - 2x^2 (= -2x^2 + xy + y^2)$

(c) $f(y,z) = y^2 + yz - 2z^2$

(d) $f(-2x,3y) = (-2x)^2 + (-2x)(3y) - 2(3y)^2 = 4x^2 - 6xy - 18y^2$

14. This is an infinite geometric series with first term 3 and constant ratio $-\frac{5}{7}$.

Its sum is $\dfrac{3}{1-\left(-\frac{5}{7}\right)} = \dfrac{3}{\frac{12}{7}} = \dfrac{21}{12} = \dfrac{7}{4}$.

[Ref: MST124 Unit 10.]

Calculus:

1. (a) $f'(x) = 4x^3 + 5 \times 3x^2 - 2x + 2$

$\qquad = 4x^3 + 15x^2 - 2x + 2$

(b) $f'(x) = \cos x$

(c) $f'(x) = 3 \times 2e^{2x} = 6e^{2x}$

(There is a table of standard derivatives in the Handbook for M208.)

[Ref: MST124 Unit 6.]

2. (a) We use the Product Rule. If $f(x) = g(x)\ h(x)$, then

$$f'(x) = g'(x)h(x) + g(x)h'(x),$$

with $g(x) = x^3 + 3$ and $h(x) = \cos x$.

So

$$\begin{aligned} f'(x) &= 3x^2 \cos x + (x^3 + 3)(-\sin x) \\ &= 3x^2 \cos x - (x^3 + 3) \sin x. \end{aligned}$$

(b) We use the Product Rule:

$$\begin{aligned} f'(x) &= e^x(x^2 + 5x - 3) + e^x(2x + 5) \\ &= e^x(x^2 + 7x + 2). \end{aligned}$$

(c) We apply the Composite Rule to the function

$$f(x) = \frac{5}{(x^2 - 1)} = 5(x^2 - 1)^{-1}.$$

So

$$\begin{aligned} f'(x) &= 5(-1)(x^2 - 1)^{-2} 2x \\ &= \frac{-10x}{(x^2 - 1)^2}. \end{aligned}$$

(d) We use the Quotient Rule.

If $f(x) = \dfrac{g(x)}{h(x)}$, then $f'(x) = \dfrac{h(x)g'(x) - h'(x)g(x)}{(h(x))^2}$.

So, since

$$\begin{aligned} f(x) &= \frac{x^2 - x + 1}{2x + 3}, \\ g(x) &= x^2 - x + 1, \\ h(x) &= 2x + 3; \end{aligned}$$

then

$$\begin{aligned} f'(x) &= \frac{(2x + 3)(2x - 1) - 2(x^2 - x + 1)}{(2x + 3)^2} \\ &= \frac{(4x^2 + 4x - 3) - (2x^2 - 2x + 2)}{(2x + 3)^2} \\ &= \frac{2x^2 + 6x - 5}{(2x + 3)^2}. \end{aligned}$$

[Ref: MST124 Unit 7.]

3. We use the Composite Rule for both parts.

(a) $f'(x) = e^{3\sin x} \times 3\cos x = 3e^{3\sin x}\cos x$

(b) $$\begin{aligned} f'(x) &= -\sin\left(3x^2+2x-6\right) \times (6x+2) \\ &= -(6x+2)\sin(3x^2+2x-6) \end{aligned}$$

[Ref: MST124 Unit 7.]

4. (a) This is an indefinite integral.

$$\begin{aligned} \int (2x^3+5)dx &= 2 \times \tfrac{1}{4}x^4 + 5x + c \\ &= \tfrac{1}{2}x^4 + 5x + c, \end{aligned}$$

where c is an arbitrary constant.

(b) This is a definite integral.

$$\begin{aligned} \int_0^{\pi/2} \cos(5t)\,dt &= \left[\tfrac{1}{5}\sin(5t)\right]_0^{\pi/2} \\ &= \tfrac{1}{5}\sin\left(\tfrac{5}{2}\pi\right) - \tfrac{1}{5}\sin(0) \\ &= \tfrac{1}{5}\times 1 - \tfrac{1}{5}\times 0 = \tfrac{1}{5} \end{aligned}$$

[Ref: MST124 Unit 7.]

5. (a) The general formula for integration by parts is

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx.$$

Let $f(x) = x$ and $g'(x) = \sin x$. Then $f'(x) = 1$ and $g(x) = -\cos x$, so

$$\begin{aligned} \int x\sin x\,dx &= -x\cos x - \int 1(-\cos x)\,dx \\ &= -x\cos x + \int \cos x\,dx \\ &= -x\cos x + \sin x + c, \end{aligned}$$

where c is an arbitrary constant.

(b) Integration by substitution uses the formula

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du,$$

where $u = g(x)$.

Let $u = 4x^3$. Then $g'(x) = 12x^2$, so

$$\begin{aligned} \int x^2 e^{4x^3}\,dx &= \frac{1}{12}\int e^{4x^3}12x^2\,dx \\ &= \frac{1}{12}\int e^u du \\ &= \frac{1}{12}e^u + c = \frac{1}{12}e^{4x^3} + c, \end{aligned}$$

where c is an arbitrary constant.

[Ref: MST124 Unit 8.]

Complex numbers:

1. A quadratic equation with these roots is $\left[z-\left(\frac{1}{5}+2i\right)\right]\left[z-\left(\frac{1}{5}-2i\right)\right]=0$; rearranging the terms inside the brackets, this is $\left[\left(z-\frac{1}{5}\right)-2i\right]\left[\left(z-\frac{1}{5}\right)+2i\right]=0$. Multiplying out this "difference of two squares" factorisation gives $\left(z-\frac{1}{5}\right)^2-(2i)^2=0$, that is, $z^2-\frac{2}{5}z+\frac{1}{25}+4=z^2-\frac{2}{5}z+\frac{101}{25}=0$. Multiplying by 25 to clear the fractions gives $25z^2-10z+101=0$, which is in the required form.

[Ref: MST124 Unit 12.]

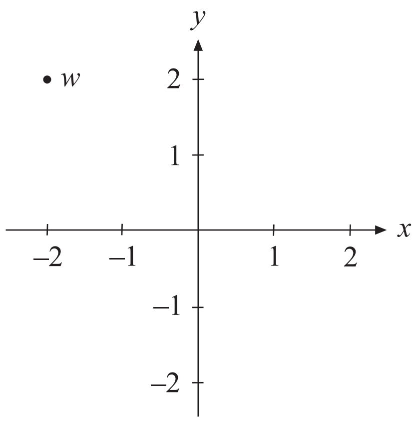
2. (a) $z+w=(3-2)+(-1+2)i=1+i$.

(b) $$\frac{z}{w}=\frac{3-i}{-2+2i}=\frac{(3-i)(-2-2i)}{(-2+2i)(-2-2i)}$$
$$=\frac{-6-6i+2i-2}{4+4}=\frac{-8-4i}{8}=-1-\frac{1}{2}i$$

(c) $|w|=\sqrt{(-2)^2+2^2}=\sqrt{8}=2\sqrt{2}$

(d) $\overline{z}=3+i$

(e) w is in the second quadrant; an Argand diagram shows that $\operatorname{Arg} w=\frac{3\pi}{4}$.



3. From Question 2(c) and (e) we have $w=2\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$, so by de Moivre's Theorem we have
$$w^6=\left(2^{(3/2)}\right)^6\left(\cos\frac{6\times 3\pi}{4}+i\sin\frac{6\times 3\pi}{4}\right)=2^9\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=512i$$

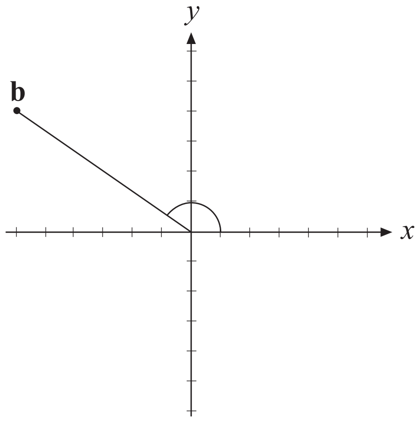
[Ref: MST124 Unit 12.]

Vectors:

1. (a) (i) (i) $\mathbf{a} - \mathbf{b} = (2 - (-6))\mathbf{i} + (3 - 4)\mathbf{j} = 8\mathbf{i} - \mathbf{j}$

(ii) $3\mathbf{a} + \mathbf{b} = (6\mathbf{i} + 9\mathbf{j}) + (-6\mathbf{i} + 4\mathbf{j}) = 13\mathbf{j}$

(iii) $|\mathbf{b}| = \sqrt{(-6)^2 + 4^2} = \sqrt{36 + 16} = \sqrt{52} = 2\sqrt{13}$; $\mathbf{b}$ is in the second quadrant, and the sine and cosine of the angle it makes with the positive x-direction are $\frac{2}{\sqrt{13}}$ and $-\frac{3}{\sqrt{13}}$ respectively.



(b) The scalar product $\mathbf{a}.\mathbf{b} = 2 \times (-6) + 3 \times 4 = -12 + 12 = 0$, so $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.

2. $\mathbf{c} = 8\left(\cos\frac{7\pi}{4}\mathbf{i} + \sin\frac{7\pi}{4}\mathbf{j}\right) = 8\left(\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}\right) = 4\sqrt{2}\mathbf{i} - 4\sqrt{2}\mathbf{j}$

[Ref: MST124 Unit 5.]

Matrices:

1. (a) (i) $$\mathbf{AB} = \begin{pmatrix} 6 & 0 \\ -2 & 3 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 4 & 2 \\ 6 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 6 \times 4 + 0 \times 6 & 6 \times 2 + 0 \times 3 \\ (-2) \times 4 + 3 \times 6 & (-2) \times 2 + 3 \times 3 \\ 0 \times 4 + 5 \times 6 & 0 \times 2 + 5 \times 3 \end{pmatrix} = \begin{pmatrix} 24 & 12 \\ 10 & 5 \\ 30 & 15 \end{pmatrix}$$

(ii) $\mathbf{BA}$ does not exist, since the number of columns of $\mathbf{B}$ is not equal to the number of rows of $\mathbf{A}$.

(iii) $2\mathbf{A} + \mathbf{B}$ does not exist, since $2\mathbf{A}$ and $\mathbf{B}$ are of different sizes.

(iv) $\mathbf{B} - \mathbf{C} = \begin{pmatrix} 4-3 & 2-2 \\ 6-1 & 3-4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 5 & -1 \end{pmatrix}$

(v) $\mathbf{A}^2$ does not exist, since $\mathbf{A}$ is not square.

(vi) $$\mathbf{B}^2 = \begin{pmatrix} 4 & 2 \\ 6 & 3 \end{pmatrix} \begin{pmatrix} 4 & 2 \\ 6 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \times 4 + 2 \times 6 & 4 \times 2 + 2 \times 3 \\ 6 \times 4 + 3 \times 6 & 6 \times 2 + 3 \times 3 \end{pmatrix} = \begin{pmatrix} 28 & 14 \\ 42 & 21 \end{pmatrix}$$

(vii) $\det \mathbf{C} = 3 \times 4 - 1 \times 2 = 12 - 2 = 10 \neq 0$, so $\mathbf{C}^{-1}$ exists:

$$\mathbf{C}^{-1} = \frac{1}{10}\begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{10} & \frac{3}{10} \end{pmatrix}$$

(viii) $\det \mathbf{B} = 4 \times 3 - 6 \times 2 = 12 - 12 = 0$, so $\mathbf{B}^{-1}$ does not exist.

(b) The system of equations in matrix form is $\mathbf{Cx} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$;
multiplying on the left by $\mathbf{C}^{-1}$ gives

$$\mathbf{x} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{10} & \frac{3}{10} \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{2}{5} \\ \frac{9}{10} \end{pmatrix}$$

So the solution is $x = \frac{2}{5}$, $y = \frac{9}{10}$.

[Ref: MST124 Unit 9.]

(c) $\operatorname{tr} \mathbf{C} = 3 + 4 = 7$ and $\det \mathbf{C} = 10$, so the characteristic equation is $\lambda^2 - 7\lambda + 10 = 0$, which factorises as $(\lambda - 2)(\lambda - 5) = 0$; so the eigenvalues are 2 and 5.

For $\lambda = 2$, the eigenvector equation is
$\begin{pmatrix} 3-2 & 2 \\ 1 & 4-2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, that is $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$;
both equations are $x + 2y = 0$, or $x = -2y$, so one eigenvector with integer components is $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

For $\lambda = 5$, the eigenvector equation is $\begin{pmatrix} 3-5 & 2 \\ 1 & 4-5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, that is $\begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; both equations reduce to $x = y$, so one eigenvector with integer components is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

[Ref: MST125 Unit 11.]

Methods of proof:

1. (a) The contrapositive of the given statement is

if n is *not* even, then n^2 is not divisible by 12

– that is,

if n is odd, then n^2 is not divisible by 12.

To prove the contrapositive, suppose that n is odd: then $n = 2k + 1$ for some integer k.

Hence $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, which is odd, since $2k^2 + 2k$ is an integer. So n^2, being odd, can't be divisible by 12, which is itself even. Hence the contrapositive is true, and hence so is the original statement.

(b) The converse of the original statement is

if n is even, then n^2 is divisible by 12.

This is false. A counterexample is given by $n = 2$: then n is even, but $n^2 = 4$ is not divisible by 12.

[Ref: MST125 Unit 9.]

2. To show that f is one-to-one, suppose that $f(x_1) = f(x_2)$ for some real numbers x_1 and x_2: then $5x_1 - 3 = 5x_2 - 3$, so (adding 3 to both sides) $5x_1 = 5x_2$, and hence (dividing both sides by 5) $x_1 = x_2$.
Hence f is one-to-one.

[Ref: MST124 Unit 3.]

3. Using proof by exhaustion:

$0^3 = 0 \equiv 0, 1^3 = 1 \equiv 1$, $2^3 = 8 \equiv 1$, $3^3 = 27 \equiv 6$, $4^3 = 64 \equiv 1$,
$5^3 = 125 \equiv 6$, $6^3 = 216 \equiv 6$ (all congruences modulo 7). Hence the congruence $x^3 \equiv 2 \pmod 7$ has no solutions.

[Ref: MST125 Units 3 and 9.]

4. (a) Aiming for a contradiction, let x be rational and y be irrational, and suppose that $x + y$ is rational. Then there are integers a and c and natural numbers (that is, strictly positive integers) b and d such that $x = \frac{a}{b}$ and $x + y = \frac{c}{d}$. Hence
$y = (x + y) - x = \frac{c}{d} - \frac{a}{b} = \frac{bc-ad}{bd}$, which is rational, since $bc - ad$ is an integer and bd a natural number. But y is irrational; so we have reached the desired contradiction, and conclude that our original supposition must be false. So the sum of a rational number and an irrational number must be irrational.

(b) Let $x = \sqrt{2}$ and $y = 2 - \sqrt{2}$. Then x is known to be irrational, and hence by (a) so is y, since it is the sum of the rational number 2 and the irrational number $-\sqrt{2}$. But $x + y = 2$, which is rational. So the sum of two irrational numbers may be rational.

[Ref: MST124 Unit 1, MST125 Unit 9.]

5. (a) Let $P(n)$ be the statement that
$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$.

When $n = 1$, $LHS = 1$ and $RHS = \frac{1}{4}.1^2.2^2 = 1$, so $P(1)$ is true.

Suppose that $P(k)$ is true for some natural number k: then
$1^3 + 2^3 + 3^3 + \ldots + k^3 = \frac{1}{4}k^2(k+1)^2$.

Hence:

$$\begin{aligned}1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 &= \frac{1}{4}k^2(k+1)^2 + (k+1)^3 \quad [\text{by } P(k)]\\ &= \frac{1}{4}(k+1)^2[k^2 + 4(k+1)]\\ &= \frac{1}{4}(k+1)^2(k^2 + 4k + 4)\\ &= \frac{1}{4}(k+1)^2(k+2)^2\end{aligned}$$

– which is $P(k+1)$.

Hence $P(k) \Rightarrow P(k+1)$ is true for $k = 1, 2\ldots$ and we conclude, by the Principle of Mathematical Induction, that $P(n)$ is true for $n = 1, 2\ldots$

(b) Let $P(n)$ be the statement that $2^n < n!$

Note that $2^1 = 2 > 1 = 1!$, $2^2 = 4 > 2 = 2!$, $2^3 = 8 > 6 = 3!$, so $P(n)$ is false for $n = 1, 2, 3$.

But $2^4 = 16 < 24 = 4!$, so $P(4)$ is true, and we have our basis.

Suppose that $P(k)$ is true for some natural number $k \geq 4$: then $2^k < k!$

Hence $2^{k+1} = 2.2^k < 2.k!$, since $P(k)$ holds, and $2.k! < (k+1)!$, since $2 < k+1$ for $k \geq 4$; so $P(k+1)$ is true.

Hence $P(k) \Rightarrow P(k+1)$ is true for $k = 4, 5 \ldots$ and we conclude, by the Principle of Mathematical Induction, that $P(n)$ is true for $n = 4, 5 \ldots$

[Ref: MST125 Unit 9.]