M337

Diagnostic quiz

Am I ready to start *Complex analysis* (M337)?

The aim of this diagnostic quiz is to help you assess how well prepared you are for M337, and to identify topics that you should revise or study before beginning the module. It is important that you are well acquainted with these topics beforehand, because parts of them will be assumed in M337, sometimes without comment. You should prepare well so that you can give your full attention to the new material that you will meet in M337.

M337 is a third-level mathematics module and as such aims to develop a higher level of mathematical maturity than that you may have required in studying second-level Open University modules in mathematics. A good pass (grade 1 or 2) in a module such as M208 *Pure mathematics*, MST210 *Mathematical methods, models and modelling* or MST224 *Mathematical methods* will serve as a solid foundation for developing these new skills.

The quiz consists of twenty-five questions split over six sections, with worked solutions at the end. The first five sections of the quiz (Algebra, Basic functions, Sketching sets and curves, Sequences and series, and Differentiation and integration) test your knowledge and facility with the fundamental mathematics that will be assumed throughout M337. The concepts illustrated in these sections will be used in the deeper parts of M337, so it is important that you are able to cope with these concepts in order to get to grips with the module. The final section (Analysis) tests your understanding of some slightly more advanced concepts. These concepts should be familiar if you have studied M208. If you have not studied M208, then some aspects of these questions may be new to you, in which case you may have to brush up on a few topics before or at the start of your studies of M337.

Solutions are provided at the end of the quiz (from page 6), along with advice and guidance (labelled 'Feedback') to help you decide whether you are ready for M337, and, if not, what you might do about it. Please read all the advice, even if you don't look at all the solutions. We suggest that you complete all twenty-five questions before reading any solutions.

If you find that you can work your way though the whole quiz within two hours, with only the occasional need to look at material you've previously studied, then you should consider yourself well prepared for M337. It is more likely that you will find there are one or two topics on which you

are a little rusty, in which case we suggest you refer to some of the resources suggested below.

After working through the quiz, you may still be unsure whether M337 is the right module for you; in that case you should visit the Study Support section of StudentHome, or consult your Student Support Team, for further advice.

Resources to help you revise for M337

All the topics of this quiz are covered in detail in M208, so if you have studied that module, then you can revise by referring to the M208 module materials.

You may have studied MST210 or MST224 already, in which case you should revise those aspects of the module materials that are relevant to M337. Many of the basic mathematical skills needed for M337 (and needed to tackle the quiz) can be found in the level one modules MST124 *Essential mathematics 1* and MST125 *Essential mathematics 2.*

Questions

You should not use a calculator to attempt the quiz. In M337 we use exact real values such as $\sqrt{2}$, π and e rather than decimals. (Note that the use of calculators is not permitted in the end-of-module examination for M337.)

Algebra

1. Simplify $3(2c(a + b) + 4) - 6(a(b + c) + 2)$.

2. Determine which of the following expressions is equal to $\dfrac{1}{a} - \dfrac{1}{b}$.

   (a) $\dfrac{1}{a - b}$ (b) $\dfrac{b - a}{ab}$ (c) $\dfrac{a - b}{ab}$ (d) $\dfrac{a + b}{a - b}$ (e) $\dfrac{b - a}{a + b}$

3. Simplify $\dfrac{1}{\sqrt{2} + 1}$.

4. Determine which of the following expressions is equal to $(a - b)^3$.
   (a) $a^3 - b^3$
   (b) $a^3 - 3a^2b + 3ab^2 - b^3$
   (c) $a^3 + 3a^2b - 3ab^2 - b^3$
   (d) $a^3 - 3a^2b + 3ab^2 + b^3$
   (e) $a^3 + 3a^2b - 3ab^2 + b^3$

5. Factorise $x^3 - 5x^2 + 6x$.

6. Find the set of all values of x for which $x(x - 3) > 0$.

7. Write $\dfrac{1}{x(x - 3)}$ in partial fractions.

Basic functions

8. Complete the following table.

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
$\sin\theta$	0		$1/\sqrt{2}$		1				
$\cos\theta$		$\sqrt{3}/2$				$-1/2$			
$\tan\theta$	0				*				

9. Show that $\sin^2(2\theta) = 4\cos^2\theta(1 - \cos^2\theta)$.

10. Find *all* values of $\theta \in [0, 2\pi)$ for which $\cos(2\theta) = -1/2$.

11. Simplify each of the following expressions. (Note that in some other modules $\log_e$ is written as ln.)

    (a) $(e^{-1})^{-1}$ (b) $\dfrac{e^{2x} \times e^{3x+1}}{e^x}$ (c) $\frac{1}{3}\log_e 8 - \frac{1}{2}\log_e 4$ (d) $e^{-\log_e 2}$

12. Find *all* real solutions of $e^{2x} + 2e^x - 3 = 0$.

    Hint: Let $y = e^x$ and solve the corresponding quadratic equation in y.

Sketching sets and curves

13. Sketch the sets $\mathbb{R} - \{n/2 : n \in \mathbb{Z}\}$ and $\{x \in \mathbb{R} : |x - 2| < 3\}$.

14. Show on a diagram the region of the plane consisting of those points for which

    $$x + y > 0 \quad \text{and} \quad x^2 + y^2 \leq 1,$$

    being careful to distinguish between those points that lie within the region and those that lie outside the region.

15. Sketch the curves given by the following equations.
    (a) $x^2 + y^2 - 4x + 2y + 1 = 0$
    (b) $x^2 + 4y^2 = 36$
    (c) $9x^2 - y^2 = 1$

Sequences and series

16. Find $\displaystyle\sum_{n=0}^{\infty} 5^{-n}$.

17. Which of the following series converge? Briefly explain your reasoning.

    (a) $1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n} + \cdots$

    (b) $1 - \dfrac{1}{2} + \dfrac{1}{3} - \cdots + \dfrac{(-1)^{n+1}}{n} + \cdots$

(c) $\quad 1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \cdots + \dfrac{1}{n^2} + \cdots$

(d) $\quad \displaystyle\sum_{n=1}^{\infty} \dfrac{5^n}{n!}$

(e) $\quad \displaystyle\sum_{n=1}^{\infty} \dfrac{n!}{10^n}$

Differentiation and integration

18. State the derivatives of the following functions.

   (a)   (i) $2x^3$ (ii) $4\sqrt{x}$ (iii) $5/x$

   (b) $\cos(1 + 5x)$

   (c) $x^3 \sin(x)$

   (d) $\dfrac{\log_e(x)}{2x}, \quad x > 0$

19. Find the derivative of $f(x) = e^{2x} \cos^2(x)$ at the point $x = \pi/2$.

20. Evaluate the following integrals.

   (a) $\displaystyle\int_0^1 x^3 \, dx$

   (b) $\displaystyle\int 12x(x^2 + 1)^5 \, dx$

   (c) $\displaystyle\int_4^5 \dfrac{1}{x(x - 3)} \, dx$

   (d) $\displaystyle\int_0^1 xe^x \, dx$

Analysis

Decide whether the following statements are true. If you believe a statement to be true, then justify why it is true. If you believe a statement to be false, then give a counterexample to demonstrate that it is false.

21. If f is a real function differentiable on the closed interval $[a, b]$, then the maximum value of f in $[a, b]$ occurs at a point $c \in [a, b]$ for which $f'(c) = 0$.

22. If f is a real function with $f(-1) = -1$ and $f(1) = 1$, then there is a point $c \in \mathbb{R}$ for which $f(c) = 0$.

23. If $\sum_{n=1}^{\infty} a_n$ is convergent then $a_n \to 0$, as $n \to \infty$.

24. Is the converse of the previous statement true?

25. Is it the case that $x < y$ if and only if $x^2 < y^2$ when

   (a) $x, y \in \mathbb{R}$,   (b)   both $x, y > 0$?

Solutions

Algebra

1.
$$3(2c(a + b) + 4) - 6(a(b + c) + 2) = 3(2ac + 2bc + 4) - 6(ab + ac + 2)$$
$$= 6ac + 6bc + 12 - 6ab - 6ac - 12$$
$$= 6bc - 6ab$$
$$= 6b(c - a)$$

2. **(b)**: $\dfrac{1}{a} - \dfrac{1}{b} = \dfrac{b}{ab} - \dfrac{a}{ab} = \dfrac{b - a}{ab}$

3. Observe that
$$\frac{1}{\sqrt{2} + 1} = \frac{1}{\sqrt{2} + 1} \times \frac{\sqrt{2} - 1}{\sqrt{2} - 1} = \frac{\sqrt{2} - 1}{2 - 1} = \sqrt{2} - 1.$$

   This trick of multiplying by one is frequently used in complex analysis.

4. **(b)**:
$$(a - b)^3 = (a - b)(a - b)^2$$
$$= (a - b)(a^2 - 2ab + b^2)$$
$$= a^3 - 2a^2b + ab^2 - a^2b + 2ab^2 - b^3$$
$$= a^3 - 3a^2b + 3ab^2 - b^3$$

   (Or use the binomial theorem.)

5. We observe that
$$x^3 - 5x^2 + 6x = x(x^2 - 5x + 6),$$

   and
$$x^2 - 5x + 6 = (x - 2)(x - 3),$$

   so
$$x^3 - 5x^2 + 6x = x(x - 2)(x - 3).$$

6. We construct a sign table for $x(x - 3)$, noting that $x(x - 3) = 0$ when $x = 0$ and $x = 3$.

	$x < 0$	$0 < x < 3$	$x > 3$
x	$-$	$+$	$+$
$(x - 3)$	$-$	$-$	$+$
$x(x - 3)$	$+$	$-$	$+$

   Hence $x(x - 3) > 0$ if (and only if) either $x < 0$ or $x > 3$. That is, $x(x - 3) > 0$ if and only if x belongs to the set
$$\{x \in \mathbb{R} : x < 0 \text{ or } x > 3\} = (-\infty, 0) \cup (3, \infty).$$

7. We need to find constants A and B such that
$$\frac{1}{x(x-3)} = \frac{A}{x} + \frac{B}{x-3}$$

for $x \neq 0, 3$. Hence, for $x \neq 0, 3$, multiplying by $x(x-3)$ gives
$$1 = A(x-3) + Bx.$$

Comparing coefficients of x and the constant term, we find that
$$A + B = 0 \text{ and } -3A = 1.$$

Hence $A = -1/3$ and $B = 1/3$, so
$$\frac{1}{x(x-3)} = \frac{1}{3}\left(\frac{1}{x-3} - \frac{1}{x}\right).$$

(Or use the cover-up method.)

Feedback Algebraic manipulations of this type appear in modules such as MST124 and MST125. You should be confident and fluent with this algebra before you start M337.

Basic functions

8.

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
$\sin\theta$	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0
$\cos\theta$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0	$-1/2$	$-1/\sqrt{2}$	$-\sqrt{3}/2$	-1
$\tan\theta$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	$*$	$-\sqrt{3}$	-1	$-1/\sqrt{3}$	0

9. As $\sin(2\theta) = 2\sin\theta\cos\theta$ we see that
$$\sin^2(2\theta) = (2\sin\theta\cos\theta)^2 = 4\sin^2\theta\cos^2\theta.$$

But $\sin^2\theta + \cos^2\theta = 1$, so
$$\sin^2(2\theta) = 4(1 - \cos^2\theta)\cos^2\theta = 4\cos^2\theta(1 - \cos^2\theta).$$

10. The equation $\cos\phi = -1/2$ has solutions $\phi = 2\pi/3 + 2m\pi$ and $\phi = 4\pi/3 + 2n\pi$, where m and n are integers. Hence $\theta = \pi/3 + m\pi$ or $\theta = 2\pi/3 + n\pi$. In the range $[0, 2\pi)$, θ has values $\pi/3$, $2\pi/3$, $4\pi/3$ and $5\pi/3$.

11. (a) $(e^{-1})^{-1} = e^{(-1)\times(-1)} = e^1 = e$

    (b) $\dfrac{e^{2x} \times e^{3x+1}}{e^x} = \dfrac{e^{2x+3x+1}}{e^x} = e^{5x+1} \times e^{-x} = e^{4x+1}$

    (c) $\frac{1}{3}\log_e 8 - \frac{1}{2}\log_e 4 = \log_e 8^{1/3} - \log_e 2 = \log_e 2 - \log_e 2 = 0$

    (d) $e^{-\log_e 2} = e^{\log_e(2^{-1})} = 2^{-1} = 1/2$

12. We make the substitution $y = e^x$ (and note that y must be positive). This gives

$$y^2 + 2y - 3 = 0,$$

which factorises to give

$$(y - 1)(y + 3) = 0.$$

Hence $y = 1$ or $y = -3$. However, y is positive, so we conclude that $y = 1$. Therefore any real solution x of $e^{2x} + 2e^x - 3 = 0$ satisfies $e^x = 1$; thus $x = 0$ is the only real solution.
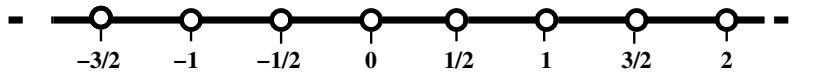

Feedback If you find that you have forgotten some of the techniques in this section, then you can revise them using materials from earlier modules such as MST124 and MST125.

Sketching sets and curves

13. As

$$\{n/2 : n \in \mathbb{Z}\} = \{\ldots, -3/2, -1, -1/2, 0, 1/2, 1, 3/2, 5/2, \ldots\}$$

(all those numbers that are integer multiples of $1/2$), the set $\mathbb{R} - \{n/2 : n \in \mathbb{Z}\}$ consists of all those real numbers that are not integer multiples of $1/2$,
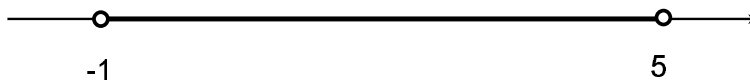

To find the set $\{x \in \mathbb{R} : |x - 2| < 3\}$, notice that $|x - 2| < 3$ if and only if

$$-3 < x - 2 < 3,$$
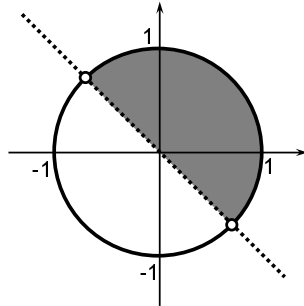
and this rearranges to give

$$-1 < x < 5.$$

Hence the set is

You can think of $\{x \in \mathbb{R} : |x - 2| < 3\}$ as being the set of all points whose distance from 2 is less than 3.

14. The inequality $x + y > 0$ rearranges to give $y > -x$ and so this is the part of the plane lying *strictly* above the line $y = -x$. The inequality $x^2 + y^2 \leq 1$ determines the region of the plane lying within the circle with centre the origin and radius 1 and includes the circle itself. The intersection of these two regions is illustrated in the following diagram. (Notice that the points where the line intersects the circle are not part of the region.)

15. (a) Here we notice that $x^2 + y^2 - 4x + 2y + 1 = 0$ can be written as $x^2 - 4x + y^2 + 2y + 1 = 0$, and then on completing the square for x and y, we obtain $(x - 2)^2 - 4 + (y + 1)^2 - 1 + 2 = 0$, which rearranges to give

$$(x - 2)^2 + (y + 1)^2 = 2^2;$$

this is the equation of a circle, centre $(2, -1)$, radius 2.
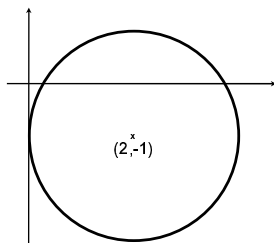
(b) In this case we divide throughout by 36 to obtain
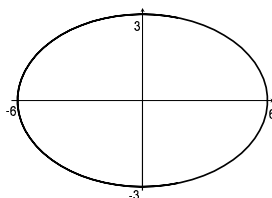
$$\frac{x^2}{6^2} + \frac{y^2}{3^2} = 1$$

which is the equation of an ellipse with centre $(0, 0)$ and with horizontal axis of length 12 and vertical axis of length 6.

(c) This is the equation of a hyperbola symmetrical about the x- and y-axes.
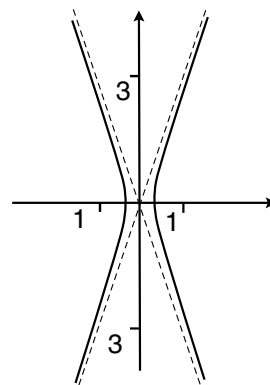
Since $9x^2 - y^2 = (3x - y)(3x + y)$, the equations of its asymptotes are given by $y = 3x$ and $y = -3x$. When $y = 0$, we see that $x = \pm 1/3$.

(a) circle   (b) ellipse   (c) hyperbola

Feedback Once again, you should revise topics from earlier modules such as MST124 and MST125 if you got stuck with some of these questions.

Sequences and series

16. This is a geometric series, hence

$$\sum_{n=0}^{\infty} 5^{-n} = \frac{1}{1 - \frac{1}{5}} = \frac{5}{4}.$$

17. (a) This series diverges: it is the harmonic series.

    (b) Since the terms of the series, $\frac{(-1)^{n+1}}{n}$, alternate in sign and $\{1/n\}$ is a null sequence, we conclude by the alternating series test that this series converges.

    (c) This is a standard example of a convergent series. It is shown in M337 that it converges to $\pi^2/6$.

    (d) We observe that if we let $a_n = 5^n/(n!)$, then

    $$\frac{a_{n+1}}{a_n} = \frac{\left(5^{n+1}/((n+1)!)\right)}{(5^n/(n!))} = \frac{5(n!)}{(n+1) \times (n!)} = \frac{5}{n+1} \to 0$$

    as $n \to \infty$. Hence, by the ratio test for series, $\sum a_n$ converges.

    (e) Since $n!/10^n \nrightarrow 0$ as $n \to \infty$ it follows that the series diverges.

Feedback The only series you are assumed to have familiarity with in M337 are geometric series, which are covered, for example, in Unit 10 of MST124. However, experience of working with real series will help you to understand M337.

Differentiation and integration

18. (a) All the parts of this question use the formula $\dfrac{d}{dx}x^n = nx^{n-1}$, where $n \in \mathbb{R}$.

    (i) $6x^2$ (ii) $2x^{-1/2} = \dfrac{2}{\sqrt{x}}$ (iii) $-5x^{-2} = -5/x^2$

    (b) $-5\sin(1 + 5x)$

    (c) $3x^2 \sin(x) + x^3 \cos(x)$

    (d) $\dfrac{2 - 2\log_e(x)}{4x^2} = \dfrac{1 - \log_e(x)}{2x^2}, \quad x > 0$

19. This is an application of the product rule for derivatives:

    $$(uv)' = u'v + uv'$$

    together with the chain rule:

    $$(f \circ g)'(x) = g'(x)(f' \circ g)(x).$$

    On letting $u(x) = e^{2x}$ and $v = \cos^2(x)$ we find from the chain rule that

    $$u'(x) = 2e^{2x} \text{ and } v'(x) = -2\sin x \cos x.$$

    Hence

    $$f'(x) = 2e^{2x}\cos^2 x - 2e^{2x}\sin x \cos x$$
    $$= 2e^{2x}\cos x(\cos x - \sin x).$$

    When $x = \pi/2$, we find $f'(x) = 0$.

20. (a) $\displaystyle\int_0^1 x^3\,dx = \left[\tfrac{1}{4}x^4\right]_{x=0}^{x=1} = \tfrac{1}{4}$

(b) On noting that $\frac{d}{dx}(x^2+1)^6 = 12x(x^2+1)^5$, we conclude that

$$\int 12x(x^2+1)\,dx = (x^2+1)^6 + C,$$

where C is an arbitrary constant.

(c) We use the partial fractions we found earlier to calculate

$$\begin{aligned}
\int_4^5 \frac{1}{x(x-3)}\,dx &= \tfrac{1}{3}\int_4^5 \frac{1}{x-3} - \frac{1}{x}\,dx \\
&= \tfrac{1}{3}\left[\log_e(x-3) - \log_e x\right]_{x=4}^{x=5} \\
&= \tfrac{1}{3}\left[\log_e \frac{x-3}{x}\right]_{x=4}^{x=5} \\
&= \tfrac{1}{3}\left(\log_e \tfrac{2}{5} - \log_e \tfrac{1}{4}\right) \\
&= \tfrac{1}{3}\log_e \tfrac{8}{5}.
\end{aligned}$$

(d) We use integration by parts, $\int f g' = fg - \int f' g$. Here we let $f(x) = x$ and $g'(x) = e^x$, and so $f'(x) = 1$ and $g(x) = e^x$. Thus

$$\begin{aligned}
\int_0^1 x e^x\,dx &= [x e^x]_0^1 - \int_0^1 1 \times e^x\,dx \\
&= e^1 - [e^x]_0^1 \\
&= e^1 - (e^1 - 1) = 1.
\end{aligned}$$

Feedback If it took you more than a few minutes to complete Question 18, then you should spend some time practising basic calculus before starting M337. Definite integrals frequently appear in the working of examples in M337, and you should be comfortable dealing with them. If the last few questions in this section took you a significant amount of time, then you should revise integration before starting M337.

Analysis

21. False. The function $f(x) = x$ satisfies $f'(x) = 1$ for all values of x, and the maximum value of f in $[0, 1]$ is $f(1) = 1$.

22. False. The function f that satisfies $f(x) = -1$ for $x \le 0$ and $f(x) = 1$ for $x > 0$ does not have the required property. If f were a continuous function, then there would be such a point c, by the Intermediate Value Theorem.

23. True. If s_N denotes the Nth partial sum of the series, that is, $s_N = \sum_{n=1}^N a_n$, then $a_N = s_N - s_{N-1}$. If the series is convergent, then the sequence of partial sums s_N is also convergent, with limit s, say. (This is the definition of convergence for series.) Hence, by the combination rules for sequences,

$$a_N = s_N - s_{N-1} \to s - s = 0 \quad \text{as} \quad N \to \infty.$$

24. False. The converse says that if $a_n \to 0$ then $\sum a_n$ is convergent. This is false as, for example, the harmonic series $1 + 1/2 + 1/3 + 1/4 + \cdots$ is divergent.

25. (a) False. The implication does not work either way: $-1 < 0$ but $(-1)^2 > 0$, and $(-1)^2 < (-2)^2$ but $-1 > -2$.

(b) True. In this case, $(y - x) > 0$ if and only if $(y^2 - x^2) = (y - x)(y + x) > 0$. So $y > x$ if and only if $y^2 > x^2$.

Feedback for students who have completed M208

If you could do most of this section without difficulty, then you probably have the mathematical maturity to study M337. At the other extreme, if you missed the point of the questions or had difficulty following the solutions, then you are going to find M337 more difficult to follow – it may be wise to consider taking a different module. If you fall in between these situations, for example if you managed a couple of the questions fine by yourself and could follow all the solutions, then you are probably suitably prepared for M337, though you may find it challenging.

Feedback for students who have not completed M208

You may not have met all the techniques needed to answer some of these questions. If you cannot answer the questions and do not understand the solutions, then you may struggle with M337 – it may be sensible to consider a different module. If you have made a reasonable attempt at some of the questions, and you can understand the solutions, then, providing that you have successfully completed the questions in other sections, you probably have enough mathematical maturity for M337, although you may have to spend time catching up on a few topics before or at the start of the module.