Are you ready for S384?

This 'Are you ready for S384?' document includes a set of self-assessment questions designed for you to test your own preparedness for studying S384 *Astrophysics of stars and exoplanets.*

Guidance

Read through this whole document carefully, work through all of the self-assessment questions and check your answers using the solutions at the end of the document. This is a useful exercise for *all* prospective students of the module. Working through this document will remind you of some of the knowledge and skills you will need to bring to S384, either from OU level 2 science and mathematics modules or from other prior study.

If you find that you can answer virtually all of the questions in this document, then it is likely that you are well prepared to take on S384. However, if you find that you have substantial difficulties with more than two questions in any of Sections 2.1, 2.2, 2.3 or 2.4, then you should take one of two actions:

(a) Consider taking one or more OU level 2 modules that will prepare you for S384, as described in Section 1.

(b) Study the S384 resources on the 'Prepare and make a head start' page of the *Physics, astronomy and planetary science* subject website (but note that this is not a substitute for taking the OU level 2 modules themselves).

1 Suggested prior study and skills

If you are intending to study S384, you should make sure you have the necessary background knowledge and skills to be able to enjoy the module fully and to give yourself the best possible chance of completing it successfully. S384 is an OU level 3 module in astrophysics that makes intellectual demands appropriate to the third year of a conventional degree. Astrophysics is a subject that relies on astronomy, physics and, most importantly, mathematics. In particular, you will be best prepared for this module if you have previously studied the appropriate level of mathematics, which includes differential and integral calculus.

Before attempting S384, you are recommended to have a good pass in OU level 2 modules covering astronomy, physics and mathematics. For example, modules recommended at Stage 2 of The Open University's R51 (BSc Physics) or Q64 (BSc Natural Sciences, Astronomy and Planetary Science specialism) degrees should give you sufficient preparation. These modules are by no means your only options. You will be able to start S384

if you have taken fairly recently, and passed well, courses equivalent to Higher National Diploma standard in physics or mathematics, or studied to at least the second year of a degree in one of these subjects.

> ### Guidance
>
> If you are coming to S384 without having studied any of the OU level 2 physical science or mathematics modules recommended above, you should establish whether your background and experience give you a sound basis on which to tackle the module. The self-assessment questions in Section 2 are designed to help you determine this.

Before studying S384, you will also find it useful to have acquired the following skills.

- Basic study skills: the ability to organise time for study, learn to pace study, read effectively to identify relevant information and data from scientific texts and accounts.

- Writing skills: the ability to write coherently, present arguments in a logical sequence, and write a scientific account with appropriate diagrams.

- Computer skills: S384 includes significant use of a personal computer, and requires access to the internet as well as online tutorials. Previous experience with using a computer is strongly recommended, especially for the manipulation and visualisation of data, either via spreadsheets and/or Python.

2   Self-assessment questions

2.1   Manipulating numbers and symbols

Mathematics is a vital tool in astrophysics – it provides the language in which ideas are expressed and in which processes are described. Before studying S384, you will therefore need to be fluent in your ability to manipulate and solve algebraic equations, including the use of powers, roots, reciprocals and imaginary numbers. You must be able to work with logarithmic ($\log_{10}$ and $\log_{e}$) and trigonometric (sin, cos, tan) functions and understand what vectors represent and how they are combined. You should also be comfortable with graphical representations of equations, and be able to interpret what they show, as well as being able to calculate and combine measurement uncertainties.

If you have difficulty with more than two of the following questions, you should consider taking an introductory mathematics module such as MST124 *Essential mathematics 1*.

Question 1

Simplify the following expression to the greatest possible extent.

$$\frac{(a^3)^{1/6} \times a^{-2}}{a^{-1/3}/a^{1/2}}.$$

Question 2

Combine the following two equations to eliminate m_2 and obtain an expression for v_2 in terms of q, i, P, m_1, G and π.

$$q = m_1/m_2,$$

$$\frac{m_1^3 \sin^3 i}{(m_1 + m_2)^2} = \frac{Pv_2^3}{2\pi G}.$$

Question 3

Find the values of x that satisfy the following quadratic equations, and then re-write each equation as the product of two factors.

(a) $6x^2 - 7x - 153 = 0$

(b) $4x^2 + 81 = 0$

Question 4

Assume $h = 6.626 \times 10^{-34}\,\mathrm{kg\,m^2\,s^{-1}}$, $G = 6.67 \times 10^{-11}\,\mathrm{kg^{-1}\,m^3\,s^{-2}}$ and $c = 2.9979 \times 10^8\,\mathrm{m\,s^{-1}}$.

(a) Determine the SI unit of the quantity defined by

$$A = \sqrt{\frac{hc}{2\pi G}}.$$

(b) Calculate the numerical value of A and express your answer in scientific notation to an appropriate number of significant figures.

Question 5

The age t, spin period P and rate of change of spin period $\dot{P}$ of a pulsar are related by the equation $t = P/(2\dot{P})$.

(a) Take the logarithm of both sides of the equation $t = P/(2\dot{P})$, and hence determine the gradient of lines of constant age on a graph of $\log_{10} \dot{P}$ versus $\log_{10} P$.

(b) What is the vertical offset (interval) between lines of constant age for every factor of ten increase or decrease in age?

Question 6

Ten independent measurements are made of the wavelength of a spectral line in the spectrum of a star. The mean value of these measurements is 585 nm and their standard deviation is 6 nm.

(a) What uncertainty should be quoted for the mean wavelength?

(b) Are the measurements consistent with a suspected true value for the wavelength of 591 nm?

(c) If the mean value needed to be known with a precision of ± 1 nm, how many measurements of the wavelength would have to be made?

Question 7

A function $y(t)$ is believed to have the form $y(t) = at^n$, where a and n are unknown constants. Given a set of pairs of data, y and t, what form of graph would you plot to enable you to determine the unknown constants?

Question 8

The shortest side of a right-angled triangle has a length of $5.0\,\text{cm}$ and the smallest internal angle is equal to 0.395 radians.

(a) Determine the sizes of all the internal angles in the triangle. Give your answers in degrees.

(b) What are the lengths of the other two sides of the triangle?

Question 9

A three-dimensional position vector $\boldsymbol{r}$ has Cartesian components $(6.0\,\text{cm}, 8.0\,\text{cm}, 10\,\text{cm})$.

(a) What are the components of the corresponding unit vector $\widehat{\boldsymbol{r}}$?

(b) What is the result of multiplying a scalar by this unit vector?

(c) What is the value of the scalar product $\boldsymbol{r} \cdot \boldsymbol{r}$?

2.2 Stars and planets

Before studying S384, you should be familiar with some of the general terminology of astronomy and planetary science. You should also be aware, in general terms, of the structure and composition of stars and planets and of how stars form and evolve. You should be familiar with observational astronomy techniques, including how stars and exoplanets are observed, and the characteristics that quantify the performance of astronomical telescopes.

If you have difficulty with more than two of these questions, you should consider taking one or more introductory astronomy modules such as S284 *Astronomy* or S283 *Planetary science and the search for life.*

Question 10

The star Regulus has a mass of $1.0 \times 10^{31}\,\text{kg}$, a radius of $2.45 \times 10^9\,\text{m}$ and a luminosity of $1.7 \times 10^{29}\,\text{W}$. Express the mass, radius and luminosity of Regulus in solar units to an appropriate number of significant figures. (You may assume that $M_\odot = 2.0 \times 10^{30}\,\text{kg}$, $R_\odot = 7.0 \times 10^8\,\text{m}$ and $L_\odot = 3.8 \times 10^{26}\,\text{W}$.)

Question 11

The star Altair has a trigonometric parallax of $\pi = 0.198''$, and a proper motion of $\mu = 0.66''$ per year.

(a) Calculate the distance of Altair in units of pc (parsec).

(b) Find the magnitude of its transverse velocity in $\text{km}\,\text{s}^{-1}$. (Note that $1\,\text{pc} = 3.086 \times 10^{13}\,\text{km}$.)

Question 12

In a few paragraphs, briefly outline the key events in the life cycle of a star and describe how the evolutionary track of a star may be represented on a Hertzsprung–Russell diagram.

Question 13

Explain how (a) the absolute magnitude and (b) the spectral type of a star may be determined experimentally, in order to place a star on a Hertzsprung–Russell diagram.

Question 14

Explain what is meant by the effective temperature of a star and state how it is related to the star's luminosity and radius.

Question 15

Consider the stars Regulus (class B7 V), Procyon (class F5 V) and Betelgeuse (class M2 I). Use the classification of each star to arrange them in order of:

(a) increasing effective temperature

(b) increasing mass

(c) increasing radius

(d) increasing luminosity.

Question 16

What is the proton–proton chain?

Question 17

Broadly speaking, what are the two types of planet present in the Solar System, and how do their structural properties compare?

Question 18

(a) Briefly describe the three main techniques by which planets around other stars have been detected.

(b) For a planet of a given mass, which methods are more likely to detect planets with small orbital radii rather than those with large orbital radii?

(c) For a planet of a given orbital radius, which methods are more likely to detect planets with large masses or radii, rather than those with small masses or radii?

(d) Aside from the obvious observational difficulties involving smaller brightness at greater distance, which methods work independently of distance?

Question 19

According to a particular catalogue, the star Deneb has coordinates RA 20 h 41 m 25.9 s and declination $+45°$ $16'$ $49''$. Why do you think the right ascension has been specified to the first decimal place of seconds but the declination is rounded to the nearest whole second of arc?

2.3   Calculus

Before studying S384, in addition to the basic mathematics described earlier, you should also be comfortable with calculus representation (e.g. dx/dt and $\int y \, dx$) and be able to manipulate and solve differentials and integrals involving simple algebraic functions, exponentials and trigonometric functions. An awareness of methods such as the chain rule, product rule, integration by substitution and integration by parts will be a distinct advantage.

If you have difficulty with more than two of these exercises, you should consider taking a mathematics module such as MST224 *Mathematical methods*.

Question 20

A graph is plotted of the speed $v(t)$ of a particle as a function of time t.

(a)   What is signified by the gradient of the graph at a particular value of t and how may this be written as a differential function?

(b)   What is signified by the area under the graph between $t = t_1$ and $t = t_2$ and how may this be written as an integral function?

Question 21

If $y(t) = 6\sin(3t^2)$, use the chain rule to find dy/dt.

Question 22

If $y(x) = 2x^3/(x+3)^4$, use the product rule to find dy/dx.

Question 23

The pressure P, volume V and temperature T in an interstellar gas cloud are related by $PV = NkT$, where N and k are constants. Use the technique of logarithmic differentiation with respect to time to find an expression for $\dot{T}/T$, where $\dot{T}$ is a shorthand for dT/dt.

Question 24

Approximate the function $f(x) = \exp(3x)$ using a second-order Maclaurin series expansion.

Question 25

If ρ is a scalar field describing the density inside a star, write down an expression for grad ρ in Cartesian coordinates and explain what it signifies.

Question 26

Evaluate the indefinite integral

$$\int \left(\frac{2}{x} + 3x^3\right) \mathrm{d}x.$$

Question 27

Using the substitution $u = x^2 - 1$, evaluate the definite integral

$$\int_0^1 (x^2 - 1)^4 \, 2x \, \mathrm{d}x.$$

Question 28

Starting from the product rule for differentiation, derive the expression which describes the technique of integration by parts. In what circumstances can this technique make integration easier?

Question 29

If $\rho(r, \phi, \theta)$ is a function describing the density inside a star with reference to spherical polar coordinates, explain what is signified by the multiple integral

$$\int_{\theta=0}^{\theta=\pi} \int_{\phi=0}^{\phi=2\pi} \int_{r=0}^{r=R} \rho(r, \phi, \theta) \, r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\phi \, \mathrm{d}\theta.$$

2.4   Physics

The physics knowledge required for S384 includes awareness of general concepts such as force, work, energy, power and momentum, and familiarity with Newton's laws of motion and of gravity. You should recognise basic concepts in electricity and magnetism, and properties of matter including gases. You should also be comfortable with the ideas of emission lines, absorption lines and continuous spectra, and the information they convey. Finally, an awareness of general concepts in quantum physics, such as photons, energy levels and wave–particle duality will be useful, as will some familiarity with nuclear processes such as radioactivity.

If you have difficulty with more than two of the questions that follow, you should consider taking a physics module such as S217 *Physics: from classical to quantum.*

Question 30

A hydrogen molecule in an interstellar cloud of gas has a mass of 3.35×10^{-27} kg and a speed of $200 \, \mathrm{m\,s^{-1}}$.

(a)   What are the molecule's translational kinetic energy and the magnitude of its linear momentum?

(b)   If the molecule undergoes an acceleration in the direction of the initial velocity of $5.00 \, \mathrm{m\,s^{-2}}$ for 10.0 s, how much work is done on the molecule?

(c) What is the magnitude of the force acting on the molecule while it is accelerating?

(d) How far does the molecule travel while it is accelerating?

Question 31

Determine the mass of the Sun, given that the orbital radius of the planet Mercury is 5.79×10^{10} m and that its planetary year lasts 88.0 Earth days. (Assume that $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}.)

Question 32

The core of a star collapses suddenly to become a neutron star, and in the process no mass is lost. Describe what happens to each of the following properties of the system as a result of the collapse.

(a) moment of inertia

(b) angular momentum

(c) angular speed

(d) rotational kinetic energy.

Question 33

Consider a fixed mass of an ideal gas.

(a) What happens to the pressure exerted by the gas if it is allowed to expand while maintaining a constant temperature?

(b) What happens to the pressure exerted by the gas if its temperature is increased while maintaining a constant volume of the gas?

(c) How do the average energy and average speed of the gas molecules alter if the absolute temperature is doubled?

Question 34

(a) A hydrogen atom makes a transition from the $n = 5$ energy level to the $n = 1$ energy level. What is the energy of the photon that is emitted?

(b) A hydrogen atom absorbs a photon of energy 1.89 eV. Between which two energy levels does it make a transition?

(c) What happens if a hydrogen atom in its ground state absorbs a photon whose energy is 15.0 eV?

Question 35

The unstable isotope of carbon represented by $^{14}_{6}$C undergoes β^--decay. Write down a balanced reaction to describe this process, indicating what nucleus is formed as a result.

Question 36

A particle of electric charge q travels in a region of uniform electric field of magnitude $\mathcal{E}$ and uniform magnetic field of magnitude B at a speed v. What are the magnitudes of the electric force and magnetic force acting on the particle and how do the directions in which these forces act differ?

Question 37

A beam of electromagnetic radiation has a frequency of $1.20 \times 10^{20}\,$Hz.

(a)   What is the wavelength of this radiation?

(b)   What is the energy, in electronvolts, of the photons of which the beam is composed?

(c)   For what temperature of black-body spectrum would the mean photon energy have this value?

(d)   Which part of the electromagnetic spectrum corresponds to radiation of this photon energy? (You may assume $c = 3.00 \times 10^8\,\mathrm{m\,s^{-1}}$, $h = 6.63 \times 10^{-34}\,\mathrm{J\,s}$, $k = 1.38 \times 10^{-23}\,\mathrm{J\,K^{-1}}$ and $1\,\mathrm{eV} = 1.60 \times 10^{-19}\,$J.)

Further resources

In addition to the resources provided on the S384 preparation website, you may find some of the following books useful.

Mathematics

Lambourne, R. and Tinker, M. (2000) *Basic mathematics for the physical sciences*. Chichester: Wiley/Open University.

Lambourne, R. and Tinker, M. (2000) *Further mathematics for the physical sciences*. Chichester: Wiley/The Open University.

Astronomy

Carroll, B.W. and Ostlie, D.A. (2017) *An introduction to modern astrophysics*. 2nd edition. Cambridge: Cambridge University Press.

Geller, R.M., Freedman, R.A. and Kaufmann, W.J. (2019) *Universe*. 11th edition. New York: W.H. Freeman.

Green, S. and Jones, M.H. (2015) *An introduction to the Sun and stars*. 2nd edition. Cambridge: Cambridge University Press/The Open University.

Rothery, D.A., McBride, N. and Gilmour, I. (2018) *An introduction to the solar system*. 3rd edition. Cambridge: Cambridge University Press/The Open University.

Rothery, D.A., Gilmour, I. and Sephton, M.A. (2018) *An introduction to astrobiology*. 3rd edition. Cambridge: Cambridge University Press/The Open University.

Snow, T.P. (1991) *The dynamic Universe: an introduction to astronomy*. 4th edition. Eagan: West Publishing Company.

Physics

Benson, H. (1998) *University physics: revised edition*. Hoboken: Wiley.

Cutnell, J.D., Johnson, K.W., Young, D., Stadler, S. (2021) *Physics*. 12th edition. Hoboken: Wiley.

Halliday, D., Resnick, R. and Walker, J. (2021) *Fundamentals of physics*. 12th edition. Hoboken: Wiley.

Solutions to questions

Solution to Question 1

$$\frac{(a^3)^{1/6} \times a^{-2}}{a^{-1/3}/a^{1/2}} = \frac{a^{3/6} \times a^{1/3}}{a^2 \times a^{-1/2}} = \frac{a^{1/2} \times a^{1/3} \times a^{1/2}}{a^2}$$

$$= a^{(1/2)+(1/3)+(1/2)-2} = a^{(3+2+3-12)/6} = a^{-4/6} = a^{-2/3}.$$

Solution to Question 2

Rearranging the first equation gives

$$m_2 = m_1/q,$$

and substituting this into the second equation leads to

$$\frac{m_1^3 \sin^3 i}{\left(m_1 + \dfrac{m_1}{q}\right)^2} = \frac{Pv_2^3}{2\pi G}.$$

Dividing both the top and bottom lines of the left-hand side of this by m_1^2 gives

$$\frac{m_1 \sin^3 i}{\left[1 + (1/q)\right]^2} = \frac{Pv_2^3}{2\pi G},$$

which may be rearranged to end up with

$$v_2 = \left(\frac{2\pi G m_1}{P\left[1 + (1/q)\right]^2}\right)^{1/3} \sin i.$$

Solution to Question 3

(a) Using the general formula for solving a quadratic equation of the form $ax^2 + bx + c = 0$, namely $x = (-b \pm \sqrt{b^2 - 4ac})/(2a)$, we have

$$x = \frac{7 \pm \sqrt{49 - \left[4 \times 6 \times (-153)\right]}}{12} = \frac{7 \pm \sqrt{3721}}{12} = \frac{7 \pm 61}{12}.$$

So

$$x = 5\tfrac{2}{3} \quad \text{or} \quad x = -4\tfrac{1}{2}.$$

The original equation may therefore be written as

$$6\left(x - 5\tfrac{2}{3}\right)\left(x + 4\tfrac{1}{2}\right) = 0 \quad \text{or} \quad (3x - 17)(2x + 9) = 0.$$

(b) The equation may be rearranged as

$$4x^2 = -81 \quad \text{or} \quad 2x = i\sqrt{81},$$

where $i = \sqrt{-1}$ is the imaginary unit.

Hence, the solutions are

$$x = \pm 9i/2,$$

so the original equation may be rewritten as

$$4(x + 9i/2)(x - 9i/2) = 0 \quad \text{or} \quad (2x + 9i)(2x - 9i) = 0.$$

Solution to Question 4

(a) Both the digit 2 and the constant π are dimensionless, i.e. they have no SI unit. So putting the units for h, c and G into the equation, the SI unit of A must be

$$\sqrt{\frac{\mathrm{kg\,m^2\,s^{-1}} \times \mathrm{m\,s^{-1}}}{\mathrm{kg^{-1}\,m^3\,s^{-2}}}} = \sqrt{\frac{\mathrm{kg\,m^3\,s^{-2}}}{\mathrm{kg^{-1}\,m^3\,s^{-2}}}}.$$

The m^3 and s^{-2} cancel on the top and bottom lines, leaving

$$\sqrt{\frac{\mathrm{kg}}{\mathrm{kg^{-1}}}} = \sqrt{\mathrm{kg^2}} = \mathrm{kg},$$

so the SI unit of A is kg.

(b) Putting in the numbers:

$$A = \sqrt{\frac{6.626 \times 10^{-34} \times 2.9979 \times 10^8}{2\pi \times 6.67 \times 10^{-11}}}\,\mathrm{kg} = 2.1771 \times 10^{-8}\,\mathrm{kg}.$$

Since the values in the question are given to varying numbers of significant figures, the answer should be given to the same accuracy as the least accurate number given, which is G with only three significant figures. The final answer is therefore

$$A = 2.18 \times 10^{-8}\,\mathrm{kg}.$$

Solution to Question 5

(a) Taking logarithms to the base 10 of the equation we obtain

$$\log_{10} t = \log_{10} P - \log_{10} 2 - \log_{10} \dot{P}.$$

This may be rearranged as

$$\log_{10} \dot{P} = \log_{10} P - \log_{10} t - \log_{10} 2.$$

So, on a graph of $\log_{10} \dot{P}$ versus $\log_{10} P$, lines of constant age will have a gradient of $+1$.

(b) For every factor of ten increase (or decrease) in the age, the vertical offset (interval) between these lines will be -1 (or $+1$) logarithmic units respectively.

Solution to Question 6

(a) The uncertainty σ_m in the mean value of n measurements is related to the standard deviation s_n of the measurements by $\sigma_m = s_n/\sqrt{n}$. So $\sigma_m = 6\,\mathrm{nm}/\sqrt{10} \approx 2\,\mathrm{nm}$. (Note that this is much smaller than the uncertainty in a single measurement, which is represented by the standard deviation of 6 nm.)

(b) The difference between the mean value (585 nm) and the suspected true value (591 nm) is 6 nm, which is three times larger than the uncertainty in the mean. Assuming that values of the means that would be obtained from many sets of ten measurements have a Gaussian distribution, then the probability of the value of the mean differing from the true value by three times the uncertainty in the mean is only 0.003. It is therefore unlikely, though possible, that the true value is 591 nm.

(c) If $\sigma_m = 1\,\text{nm}$ and $s_n = 6\,\text{nm}$, then $\sqrt{n} = s_n/\sigma_m = 6$ and $n = 36$. So reducing the uncertainty from $2\,\text{nm}$ to $1\,\text{nm}$ would require almost four times as many measurements.

Solution to Question 7

Taking the logarithm of each side of the proposed equation yields

$$\log y = \log a + n \log t.$$

Therefore, plotting a graph of $\log y$ against $\log t$ would yield a straight line whose intercept on the vertical axis is equal to $\log a$ and whose gradient is equal to n. Hence, both a and n could be determined from the graph.

Solution to Question 8

(a) Since π radians is equal to $180°$, the angle in question is

$$0.395 \text{ radians} \times \frac{180°}{\pi \text{ radians}} = 22.6°.$$

One of the other internal angles must be $90°$ (a right angle).

Since the internal angles of a triangle add up to $180°$, the third angle is

$$180° - 90° - 22.6° = 67.4°.$$

(b) Let the hypotenuse of the triangle be of length h and the other unknown side be of length b. The smallest angle will be opposite the smallest side of the triangle, so $\sin 22.6° = 5.0\,\text{cm}/h$ and therefore

$$h = 13.0 \text{ cm}.$$

Similarly, $\tan 22.6° = 5.0\,\text{cm}/b$, so

$$b = 12.0 \text{ cm}.$$

Solution to Question 9

(a) A unit vector is defined as $\widehat{r} = r/r$ (i.e. a vector r divided by its own magnitude $r = |r|$), so the unit vector is dimensionless.

The magnitude of r is

$$r = |r| = \sqrt{(6.0\,\text{cm})^2 + (8.0\,\text{cm})^2 + (10\,\text{cm})^2} = \sqrt{200}\,\text{cm}.$$

So the unit vector has components

$$\widehat{r} = \left(\frac{6.0\,\text{cm}}{\sqrt{200}\,\text{cm}}, \frac{8.0\,\text{cm}}{\sqrt{200}\,\text{cm}}, \frac{10\,\text{cm}}{\sqrt{200}\,\text{cm}} \right) = (0.42, 0.57, 0.71).$$

(b) The effect of multiplying a scalar by this unit vector is to produce a vector whose magnitude is equal to that of the original scalar, and whose direction is the same as that of the vector r.

(c) The scalar product $r \cdot r$ is defined as $r^2 \cos\theta$ where, in this case, $\theta = 0°$ and so $\cos\theta = 1$. Alternatively $r \cdot r = (r_x^2 + r_y^2 + r_z^2)$. In either case, the value of the scalar product is equal to the magnitude of the vector r squared or

$$r \cdot r = |r|^2 = (6.0\,\text{cm})^2 + (8.0\,\text{cm})^2 + (10.0\,\text{cm})^2 = 200\,\text{cm}^2.$$

Solution to Question 10

All answers should be expressed to an accuracy of two significant figures.

The mass of Regulus is $(1.0 \times 10^{31}\,\text{kg})/(2.0 \times 10^{30}\,\text{kg}\,\text{M}_\odot^{-1}) = 5.0\,\text{M}_\odot$.

The radius of Regulus is $(2.45 \times 10^9\,\text{m})/(7.0 \times 10^8\,\text{m}\,\text{R}_\odot^{-1}) = 3.5\,\text{R}_\odot$.

The luminosity of Regulus is $(1.7 \times 10^{29}\,\text{W})/(3.8 \times 10^{26}\,\text{W}\,\text{L}_\odot^{-1}) = 450\,\text{L}_\odot$.

Solution to Question 11

(a) Distance in parsec is related to trigonometric parallax by $d/\text{pc} = 1/(\pi/\text{arcsec})$. So in this case,

$$d = (1/0.198)\,\text{pc} = 5.05\,\text{pc}.$$

(b) An angular shift of $0.66''$ at a distance of $5.05\,\text{pc}$ corresponds to a displacement of

$$(5.05\,\text{pc}) \times \tan(0.66/3600)^\circ = 1.616 \times 10^{-5}\,\text{pc}.$$

Converting this to a path length in kilometres yields

$$(1.616 \times 10^{-5}\,\text{pc}) \times (3.086 \times 10^{13}\,\text{km}\,\text{pc}^{-1}) = 4.987 \times 10^8\,\text{km}.$$

This is the transverse distance travelled in one year, so the magnitude of the transverse velocity is

$$\frac{4.987 \times 10^8\ \text{km/year}}{365\ \text{days/year} \times 24\ \text{hours/day} \times 3600\ \text{seconds/hour}} = 16\,\text{km}\,\text{s}^{-1}.$$

Solution to Question 12

Stars are formed by the collapse of fragments of dense molecular clouds. When the central regions become hot enough for nuclear fusion to be initiated, the star is born on the zero age main sequence of the Hertzsprung–Russell diagram. The star remains on the main sequence while undergoing hydrogen fusion in its core by the proton–proton chain or the CNO cycle.

When hydrogen in the core is exhausted, helium fusion may begin, and occurs by the triple alpha process. In low-mass stars this happens by way of an explosive helium flash. Other nuclear fusion reactions are subsequently possible in massive stars.

When nuclear fuel is exhausted, the star ends its life in one of several ways, depending on its mass. Low mass stars shed their outer layers as planetary nebulae and the core collapses to form a white dwarf. Massive stars explode as supernovae and the core collapses to form a neutron star or black hole.

A star's position on the Hertzsprung–Russell diagram is determined by its luminosity (vertical axis) and photospheric temperature (horizontal axis). For the majority of its life, a star will remain in almost the same place on the Hertzsprung–Russell diagram, i.e. on the main sequence. As it undergoes different nuclear fusion reactions in its core, its luminosity and temperature will change, so it will move around the Hertzsprung–Russell diagram. The particular track followed depends essentially on the mass of the star.

Solution to Question 13

(a) A star's absolute magnitude M may be calculated by measuring its apparent magnitude m, and then correcting for its distance away d and the interstellar absorption that the light has suffered A. The relationship is

$$M = m + 5 - 5\log_{10} d + A,$$

where d is in parsec.

(b) A star's spectral type is related to its photospheric temperature. The colour of the star may be determined as the difference between two photometric magnitudes, such as $m_B - m_R$; redder stars (with a larger value of $m_B - m_R$) are cooler while bluer stars (with a smaller value of $m_B - m_R$) are hotter. A more detailed measure of the temperature, and hence spectral type, may be obtained from measuring the spectral lines from the star's photosphere:

- O-type stars have lines from ionised and neutral helium with weak hydrogen lines
- B-type stars have weaker ionised helium lines, stronger neutral helium lines and stronger hydrogen lines
- A-type stars have the strongest hydrogen lines and no helium lines
- in F-type stars, hydrogen lines are weaker but ionised metal lines (such as calcium and iron) appear
- in G-type stars metal lines are stronger and hydrogen lines are very weak
- K-type stars have the strongest calcium lines
- the coolest M-type stars have molecular bands, notably those of titanium oxide.

Solution to Question 14

The effective temperature T_{eff} of a star is the temperature T of a black-body source that has the same radius and luminosity as the star. The flux F escaping through the star's surface is

$$F = \frac{L}{4\pi R^2},$$

where L is the star's luminosity and R is the star's radius. However, the flux is related to the black-body temperature by the Stefan–Boltzmann law:

$$F = \sigma T^4,$$

where $\sigma = 5.67 \times 10^{-8}\,\text{W}\,\text{m}^{-2}\,\text{K}^{-4}$ is the Stefan–Boltzmann constant. Therefore, the effective temperature of a star may be *defined* as

$$T_{\text{eff}} = \left(\frac{L}{4\pi\sigma R^2}\right)^{1/4}.$$

Solution to Question 15

From the classifications, Regulus is a relatively hot main-sequence star, Procyon is a cooler main-sequence star and Betelgeuse is an even cooler supergiant star.

(a) In order of increasing effective temperature: Betelgeuse, Procyon, Regulus.

(b) In order of increasing mass: Procyon, Regulus, Betelgeuse.

(c) In order of increasing radius: Procyon, Regulus, Betelgeuse.

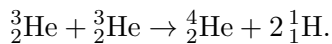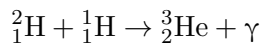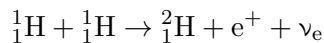(d) In order of increasing luminosity: Procyon, Regulus, Betelgeuse.

The effective temperature T_{eff}, mass M, radius R and luminosity L of the three stars (in solar units) are as shown in Table 1.

Table 1   Selected properties of Betelgeuse Procyon and Regulus.

star (classification)	T_{eff}/K	$M/\text{M}_\odot$	$R/\text{R}_\odot$	$L/\text{L}_\odot$
Betelgeuse (M2 I)	3500	20	800	50 000
Procyon (F5 V)	6500	1.3	1.2	2.5
Regulus (B7 V)	14 500	5.0	3.5	450

Solution to Question 16

The proton–proton (p–p) chain is the route by which low-mass stars fuse hydrogen nuclei (protons) to make nuclei of helium. The most common route for the p–p chain is as follows:

$$_1^1\text{H} + {_1^1}\text{H} \rightarrow {_1^2}\text{H} + \text{e}^+ + \nu_\text{e}$$

$$_1^2\text{H} + {_1^1}\text{H} \rightarrow {_2^3}\text{He} + \gamma$$

$$_2^3\text{He} + {_2^3}\text{He} \rightarrow {_2^4}\text{He} + 2\,{_1^1}\text{H}.$$

(About 25 MeV of energy is released for each helium nucleus formed in this way.)

Solution to Question 17

The Solar System is broadly separated into terrestrial planets (predominantly rocky and metallic in composition) in the inner region, and giant planets (predominantly gaseous) in the outer regions. The terrestrial planets have metallic cores overlaid by rocky mantles; some of them have atmospheres. The giant planets have small rocky cores, overlaid by either a thick layer of fluid helium and metallic hydrogen (Jupiter and Saturn) or liquid icy materials including water, ammonia and methane (Uranus and Neptune). The outermost layer of a gas giant is a thick atmosphere composed predominantly of hydrogen and helium.

Solution to Question 18

(a) The three main techniques are as follows: (i) the Doppler spectroscopy or radial velocity technique whereby the periodic shift in spectral lines from a star indicates it is being tugged to-and-fro by an orbiting planet; (ii) the transit photometry or occultation technique whereby a periodic dip in the brightness of a star indicates the transit of an orbiting planet; (iii) the gravitational microlensing technique whereby the rapid brightening of a background star undergoing microlensing has a small spike superimposed on its light curve indicating the presence of a planet in the vicinity of the foreground star.

(b) A planet that is closer to its star will have a shorter orbital period. A planet with a small orbital radius is more likely to pass across the disc of the star, and will do so more often, than a planet with a larger orbital radius. Such a planet will also be more likely to give rise to a microlensing event which is close in time with the microlensing event due to its parent star. Close-in planets also have greater speeds and so can exhibit greater radial velocity Doppler shifts. Hence, all three techniques are more likely to detect planets with small orbital radii rather than those with large orbital radii.

(c) Although the densities of planets do vary, it is fair to say that larger planets will generally also be more massive. Such large planets will block out more light from their parent star when they transit. Planets with large mass will have a larger Einstein ring and so will be more likely to microlens a background star, and will also have a longer microlensing timescale. A more massive planet will also cause a greater radial velocity Doppler shift, and so result in a more easily detectable wavelength shift. Hence, all three techniques are likely to detect planets with large radii and masses rather than those with small radii and masses.

(d) Neglecting the fact that brightness will fall with distance, all three techniques are independent of the distance to the planetary system.

Solution to Question 19

At the celestial equator, $1\,\text{s}$ of right ascension is 15 times larger than $1''$. At the declination of Deneb, the multiple is reduced by a factor $\cos 45°16'49'' = 0.70$, so $1\,\text{s}$ of right ascension is $0.70 \times 15 = 10.5$ times larger than $1''$ of declination. By writing the declination to the nearest arcsecond, the implied accuracy of the position is $\pm 0.5''$ in this coordinate. If the right ascension is known to the same accuracy, this corresponds to $\pm 0.5/(10.5\,\text{s}^{-1}) \approx 0.05\,\text{s}$ of right ascension at this declination. Hence, if both coordinates are measured to the same accuracy, it makes sense to quote the right ascension to the nearest $0.1\,\text{s}$.

Solution to Question 20

(a) The gradient of a graph of speed against time at a particular value of t is the magnitude of the acceleration (or the negative of the magnitude if the gradient is negative, i.e. a deceleration) of the particle at that instant of time. In symbols,

$$a(t) = \frac{\mathrm{d}v(t)}{\mathrm{d}t}.$$

(b) The area under a graph of speed against time between two limits is the distance s covered by the particle between these two times. In symbols,

$$s = \int_{t_1}^{t_2} v(t)\,\mathrm{d}t.$$

Solution to Question 21

Put $u = 3t^2$ then

$$\frac{\mathrm{d}u}{\mathrm{d}t} = 6t.$$

Also, $y = 6\sin u$, so

$$\frac{\mathrm{d}y}{\mathrm{d}u} = 6\cos u = 6\cos(3t^2).$$

Now, using the chain rule,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}t},$$

we have

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 6\cos(3t^2) \times 6t = 36t\cos(3t^2).$$

Solution to Question 22

First, put $u = 2x^3$ and $v = (x+3)^{-4}$, then $y = uv$ and we can use the product rule:

$$\frac{\mathrm{d}(uv)}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}.$$

Therefore, $\mathrm{d}u/\mathrm{d}x = 6x^2$ and $\mathrm{d}v/\mathrm{d}x = -4(x+3)^{-5}$. Finally putting this all together,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \left[2x^3 \times -4\,(x+3)^{-5}\right] + \left[(x+3)^{-4} \times 6x^2\right]$$
$$= \frac{-8x^3}{(x+3)^5} + \frac{6x^2}{(x+3)^4}.$$

Solution to Question 23

Taking natural logarithms of each side of the equation,

$$\log_e P + \log_e V = \log_e Nk + \log_e T.$$

Now taking the time derivative of this expression, noting that in general $\mathrm{d}(\log_e x)/\mathrm{d}t = \dot{x}/x$, and that N and k are constants, we have

$$\frac{\dot{P}}{P} + \frac{\dot{V}}{V} = \frac{\dot{T}}{T}.$$

Solution to Question 24

The first and second derivatives of the function are $f'(x) = 3\exp(3x)$ and $f''(x) = 9\exp(3x)$. So the second-order Maclaurin series expansion is

$$\exp(3x) = f(0) + xf'(0) + x^2 f''(0)/2!$$

$$\exp(3x) = \exp 0 + x(3\exp 0) + x^2(9\exp 0)/2$$

$$\exp(3x) = 1 + 3x + 9x^2/2.$$

Solution to Question 25

In terms of Cartesian components,

$$\text{grad}\,\rho = \nabla\rho = \left(\frac{\partial\rho}{\partial x}, \frac{\partial\rho}{\partial y}, \frac{\partial\rho}{\partial z}\right).$$

The quantity $\text{grad}\,\rho$ is a vector field which describes the density gradient within the star.

Solution to Question 26

$$\int \left(\frac{2}{x} + 3x^3\right) \mathrm{d}x = 2\log_e x + \frac{3x^4}{4} + C.$$

Solution to Question 27

Setting $u = x^2 - 1$, then $\mathrm{d}u/\mathrm{d}x = 2x$ or $\mathrm{d}u = 2x\,\mathrm{d}x$. Substituting these into the original integral gives

$$\int_{x=0}^{x=1} \left(x^2 - 1\right)^4 2x\,\mathrm{d}x = \int_{u=-1}^{u=0} u^4\,\mathrm{d}u.$$

Now, evaluating this integral:

$$\int_{-1}^{0} u^4\,\mathrm{d}u = \left[\frac{u^5}{5}\right]_{-1}^{1} = 0 - (-1)^5/5 = 1/5.$$

Solution to Question 28

The product rule for differentiation is

$$\frac{\mathrm{d}(uv)}{\mathrm{d}t} = u\frac{\mathrm{d}v}{\mathrm{d}t} + v\frac{\mathrm{d}u}{\mathrm{d}t}$$

where u and v are both functions of t in this case. If we now *integrate* the above expression with respect to t we obtain

$$\int \frac{\mathrm{d}(uv)}{\mathrm{d}t}\,\mathrm{d}t = \int u\frac{\mathrm{d}v}{\mathrm{d}t}\,\mathrm{d}t + \int v\frac{\mathrm{d}u}{\mathrm{d}t}\,\mathrm{d}t$$

remembering that $\mathrm{d}t/\mathrm{d}t = 1$, this simplifies to

$$uv = \int u\,\mathrm{d}v + \int v\,\mathrm{d}u$$

(ignoring the formal integration constant, which will vanish when the integral is evaluated between two limits.)

Finally, rearranging this result we obtain the desired expression for integration by parts,

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u.$$

Success with the formula relies on choosing u and $\mathrm{d}v$ such that $\int v \,\mathrm{d}u$ is easier to calculate than $\int u \,\mathrm{d}v$.

Solution to Question 29

This is a volume integral and the evaluation of it will give the total mass of the star.

Solution to Question 30

(a) The translational kinetic energy of the molecule is

$$E_{\mathrm{KE}} = 0.5mv^2$$
$$= 0.5 \times (3.35 \times 10^{-27}\,\mathrm{kg}) \times (200\,\mathrm{m\,s^{-1}})^2$$
$$= 6.70 \times 10^{-23}\,\mathrm{J}.$$

The magnitude of the linear momentum of the molecule is

$$p = mv$$
$$= (3.35 \times 10^{-27}\,\mathrm{kg}) \times (200\,\mathrm{m\,s^{-1}})$$
$$= 6.70 \times 10^{-25}\,\mathrm{kg\,m\,s^{-1}}.$$

(b) The work done on the molecule is equal to its change in kinetic energy (or equivalently the magnitude of the applied force multiplied by the distance over which it acts). The final speed of the molecule is found from

$$v = u + at$$
$$= (200\,\mathrm{m\,s^{-1}}) + (5.00\,\mathrm{m\,s^{-2}}) \times (10.0\,\mathrm{s})$$
$$= 250\,\mathrm{m\,s^{-1}}.$$

The final translational kinetic energy is

$$E_{\mathrm{KE}} = 0.5mv^2$$
$$= 0.5 \times (3.35 \times 10^{-27}\,\mathrm{kg}) \times (250\,\mathrm{m\,s^{-1}})^2$$
$$= 10.5 \times 10^{-23}\,\mathrm{J}.$$

So the work done on the molecule is

$$W = \Delta E_{\mathrm{KE}}$$
$$= (10.47 - 6.70) \times 10^{-23}\,\mathrm{J}$$
$$= 3.77 \times 10^{-23}\,\mathrm{J}.$$

(c) Using a form of Newton's second law of motion, $F = ma$, the magnitude of the force acting on the molecule is

$$F = (3.35 \times 10^{-27}\,\mathrm{kg}) \times (5.00\,\mathrm{m\,s^{-2}})$$
$$= 1.68 \times 10^{-26}\,\mathrm{N}.$$

(d) The distance covered by the molecule while it is undergoing an acceleration may be found from

$$s = ut + \frac{1}{2}at^2$$

$$= (200 \, \text{m s}^{-1} \times 10.0 \, \text{s}) + \left[\frac{1}{2} \times 5.00 \, \text{m s}^{-2} \times (10.0 \, \text{s})^2\right]$$

$$= 2250 \, \text{m} \quad \text{or} \quad 2.25 \, \text{km}.$$

Solution to Question 31

A planet of mass m moving in a circular orbit of radius r with uniform angular speed ω must be subject to a centripetal force of magnitude $F = mr\omega^2$. If this force is supplied by the gravitational attraction of the Sun, then by Newton's law of universal gravitation,

$$mr\omega^2 = \frac{GM_\odot m}{r^2}.$$

For a planet with angular speed ω, the orbital period is $P_{\text{orb}} = 2\pi/\omega$, so replacing ω in the equation above by $2\pi/P_{\text{orb}}$ gives

$$\frac{(2\pi)^2 mr}{P_{\text{orb}}^2} = \frac{GM_\odot m}{r^2}.$$

Rearranging and cancelling the common terms, this yields

$$M_\odot = \frac{(2\pi)^2 r^3}{GP_{\text{orb}}^2}.$$

Putting in the numbers we have,

$$M_\odot = \frac{(2\pi)^2 \times (5.79 \times 10^{10} \, \text{m})^3}{(6.67 \times 10^{-11} \, \text{N m}^2 \, \text{kg}^{-2}) \times (88.0 \times 24 \times 3600 \, \text{s})^2}$$

$$= 1.99 \times 10^{30} \, \text{kg}.$$

Solution to Question 32

(a) The moment of inertia I of a solid sphere is proportional to its radius squared (in fact, for a uniform sphere $I = 2MR^2/5$). So, since the mass M remains constant but the radius R is reduced, the moment of inertia will *decrease*.

(b) Since no external torques act, the angular momentum of the system is *constant* ($\Gamma = \text{d}J/\text{d}t = 0$).

(c) Since the magnitude of the angular momentum J depends on the product of the moment of inertia and ω, the angular speed ($J = I\omega$), the angular speed must *increase* to compensate for the decrease in the moment of inertia.

(d) The rotational kinetic energy depends on the moment of inertia and the square of the angular speed ($E_{\text{rot}} = \frac{1}{2}I\omega^2$), but $I\omega$ is constant and ω increases, so the rotational kinetic energy must also *increase*.

Solution to Question 33

(a) Using the ideal gas equation, $PV = NkT$, if V increases while N and T are kept fixed, then P must decrease. So the pressure exerted by the gas will decrease.

(b) Using the ideal gas equation again, if T increases while N and V are kept fixed, then P must increase. So the pressure exerted by the gas will increase.

(c) The average energy of the molecules is proportional to the absolute temperature and the average speed of the molecules is proportional to the square root of the absolute temperature. So if the temperature is doubled, the average energy also doubles, whereas the average speed increases by a factor of $\sqrt{2}$.
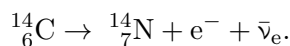
Solution to Question 34

The energy levels of the hydrogen atom are determined by the equation $E_n = -13.60\,\text{eV}/n^2$, where n is an integer. The energies of the first six energy levels are therefore: $E_1 = -13.60\,\text{eV}$, $E_2 = -3.40\,\text{eV}$, $E_3 = -1.51\,\text{eV}$, $E_4 = -0.85\,\text{eV}$, $E_5 = -0.54\,\text{eV}$ and $E_6 = -0.38\,\text{eV}$.

(a) In making a transition from the $n = 5$ energy level to the $n = 1$ energy level, the atom emits a photon of energy

$$(-0.54\,\text{eV}) - (-13.60\,\text{eV}) = 13.06\,\text{eV}.$$

(b) In order to absorb a photon of energy $1.89\,\text{eV}$, a hydrogen atom must make a transition from the $n = 2$ energy level to the $n = 3$ energy level.

(c) The atom is ionised. The first $13.6\,\text{eV}$ is used to raise the atom from the ground-state energy level to a state in which the nucleus (proton) and electron are widely separated. The remaining $1.4\,\text{eV}$ is transferred to the proton and electron as kinetic energy.

Solution to Question 35

The process of β^--decay occurs when a neutron transforms into a proton. So the mass number of the nucleus remains the same, but its atomic number increases by one. The balanced reaction is

$$^{14}_{6}\text{C} \rightarrow {}^{14}_{7}\text{N} + \text{e}^- + \bar{\nu}_e.$$

The resulting nucleus is an isotope of nitrogen and an electron (β^--particle) and an electron antineutrino are emitted, carrying away the released energy.

Solution to Question 36

The magnitude of the electric force is $F_{\text{el}} = q\mathcal{E}$ and the magnitude of the magnetic force is $F_{\text{mag}} = qvB\sin\theta$, where θ is the angle between the velocity vector of the particle and the direction of the magnetic field. The direction of the electric force vector is parallel to the direction of the electric field, while the magnetic force acts in a direction which is at right angles to the plane containing the velocity vector of the particle and the magnetic field vector.

Solution to Question 37

(a) Using $c = \lambda\nu$, the wavelength of the radiation is

$$\lambda = \frac{3.00 \times 10^8 \, \text{m s}^{-1}}{1.20 \times 10^{20} \, \text{Hz}}$$

$$= 2.50 \times 10^{-12} \, \text{m} \quad \text{or} \quad 0.00250 \, \text{nm}.$$

(b) Using $E_\text{ph} = h\nu$, the energy of the photons of which this radiation is composed is

$$E_\text{ph} = (6.63 \times 10^{-34} \, \text{J s}) \times (1.20 \times 10^{20} \, \text{Hz})$$

$$= 7.96 \times 10^{-14} \, \text{J}.$$

Converting this into electronvolts:

$$E_\text{ph} = \frac{7.96 \times 10^{-14} \, \text{J}}{1.60 \times 10^{-19} \, \text{J eV}^{-1}}$$

$$= 4.98 \times 10^5 \, \text{eV} \quad \text{or about } 500 \, \text{keV}.$$

(c) Using $\langle E_\text{ph} \rangle = 2.70 k T$, a black-body spectrum whose mean photon energy is $500 \, \text{keV}$ would have a temperature of

$$T = \frac{\langle E_\text{ph} \rangle}{2.70 k}$$

$$= \frac{7.96 \times 10^{-14} \, \text{J}}{2.70 \times 1.38 \times 10^{-23} \, \text{J K}^{-1}}$$

$$= 2.14 \times 10^9 \, \text{K}$$

or just over two billion kelvin.

(d) Photons with this energy lie close to the boundary between the X-ray and gamma-ray parts of the electromagnetic spectrum.