Faculty of Science, Technology, Engineering and Mathematics
MST368 Graphs, games and designs

Diagnostic quiz

Am I ready to start MST368?

The questions that follow are designed to help you answer this.

Being an inter-disciplinary module, MST368 aims to be accessible to any students who have studied some mathematics at Level 1. It is therefore not as demanding in terms of mathematical prerequisites as would be the case for other Level 3 mathematics modules.

The module shows you how to use relatively simple mathematical ideas and processes to model a variety of naturally occurring problems, and obtain solutions that are sometimes the best possible, but are at any rate better than you could obtain without the methods described. The most important thing that you need in order to tackle this course is a willingness to get involved both with the mathematical ideas and with their application.

The questions are divided into two sections. The first deals with basic mathematical skills and techniques that you should have met in your previous studies. The second is a set of problems that appear in the module, and give a flavour of the types of problems with which the module involves you.

If you find Part 1 difficult, then you may wish to consider taking an Open University mathematics module such as MST124 before proceeding to MST368.

As far as Part 2 is concerned, do not be put off if you cannot answer all of these problems; the question is, did you find them interesting? If so, and if you are happy with the ideas of Part 1, then you are likely to enjoy studying MST368.

Do contact your Student Support Team via StudentHome if you have any queries about MST368.

Part 1

Question 1.1

Find:

(a) $2^6 \times 2^{-3}$ (b) $2^n \div 2^3$ (c) $k^4 \times k^7 \times (k^2)^{-1}$.

Question 1.2

Simplify the following inequalities:

(a) $2(x+2) \leq 5x+1$;

(b) $\dfrac{1}{x} - \dfrac{1}{4} < 0$ (under the assumption that x is positive).

Question 1.3

Consider the matrices

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 3 & 2 \\ 0 & 2 & 1 & 7 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 2 & -2 \\ 1 & 3 & 1 \\ 2 & 1 & 1 \\ 3 & -3 & 1 \end{bmatrix}.$$

(a) Calculate the matrix product $\mathbf{AB}$.

(b) Why does the matrix product $\mathbf{BA}$ not make sense?

Question 1.4

Matrix arithmetic can be performed modulo 2; that is, using just the numbers 0 and 1 as entries, where calculations are carried out as follows:

$$\begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array} \qquad \begin{array}{c|cc} \times & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}$$

So, for example, in arithmetic modulo 2, we have $1+1=0$ and $1 \times 0 = 0$.

Let $\mathbf{C} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$, $\mathbf{C}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{C}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

Find the products $\mathbf{CC}_1$ and $\mathbf{CC}_2$, with matrix arithmetic performed modulo 2.

Question 1.5

The following are the coordinates of four of the corners of the unit cube:

$$(0,0,0), (0,0,1), (0,1,1) \text{ and } (1,1,0).$$

(a) What are the coordinates of the remaining corners?

(b) How many edges does the cube have?

Question 1.6

Given that $b = v = 7$ and $k = 3$, and that the equations $bk = vr$ and $\lambda(v - 1) = r(k - 1)$ hold, find the value of λ.

Question 1.7

A line ℓ passes through the points $(-4, 2)$ and $(2, 5)$.

(a) Find the equation of ℓ in the form $y = mx + c$.

(b) Does the point $(-1, 3)$ belong to ℓ?

Question 1.8

A recurrence relation has

$$u_1 = 1, u_2 = 2 \text{ and}$$

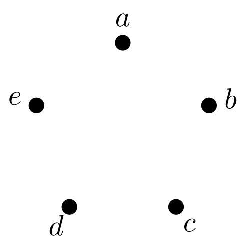
$$u_n = 2u_{n-1} + (u_1u_{n-2} + u_2u_{n-3} + \cdots + u_{n-3}u_2 + u_{n-2}u_1).$$

So $u_3 = 2u_2 + (u_1u_1)$ and $u_4 = 2u_3 + (u_1u_2 + u_2u_1)$.

Find the values of u_5 and u_6.

Question 1.9

Consider the following five dots.



How many different triangles are there with each corner at one of the dots? What is the answer for n dots? Explain briefly.

Question 1.10

You are told:

> If a student has successfully completed MST124, then they are in a good position to study MST368.

Which of the following deductions are valid?

(a) Ayesha has successfully completed MST124, so she is in a good position to study MST368.

(b) Beata is in a good position to study MST368, so she has successfully completed MST124.

(c) Connor has not successfully completed MST124, so he is not in a good position to study MST368.

Question 1.11

Prove the following statement is true.

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2 \text{ for all } n \in \mathbb{N}.$$

(You may prefer to use proof by induction, or to prove it directly.)

Part 2

Question 2.1 – Map Colouring

Consider the following map of the USA (excluding Alaska and Hawaii):

It is common for maps of this kind to be coloured in such a way that states (or countries) that share a common boundary line are coloured differently. This enables us to distinguish easily between the various states, and to locate the state boundaries. The question arises:

> How many colours are needed to colour the entire map?

One might reasonably expect that the larger and more complicated a map, the more colours we might need to colour it (though, actually, this turns out not to be true).

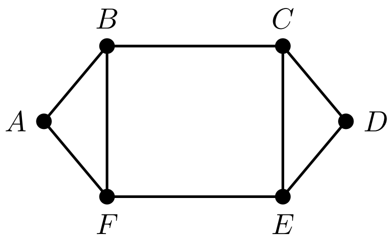
Can the above map of the USA be coloured with just three colours?

Hint: Consider Nevada (shaded) and its neighbouring states.

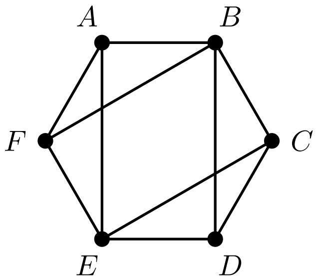
Question 2.2 – Connection problems

Much of MST368 is concerned with structures called *graphs*. These are *not* the plots of y against x with which you may well be familiar; graphs in MST368 are structures consisting of dots, some pairs of which are joined by lines. These graphs may represent a wide variety of practical situations in which we have a certain number of objects some pairs of which are linked in some way.

For example, here are two graphs that could represent simple telephone exchange networks, where some pairs of exchanges are *linked* directly by a cable.



(1)



(2)

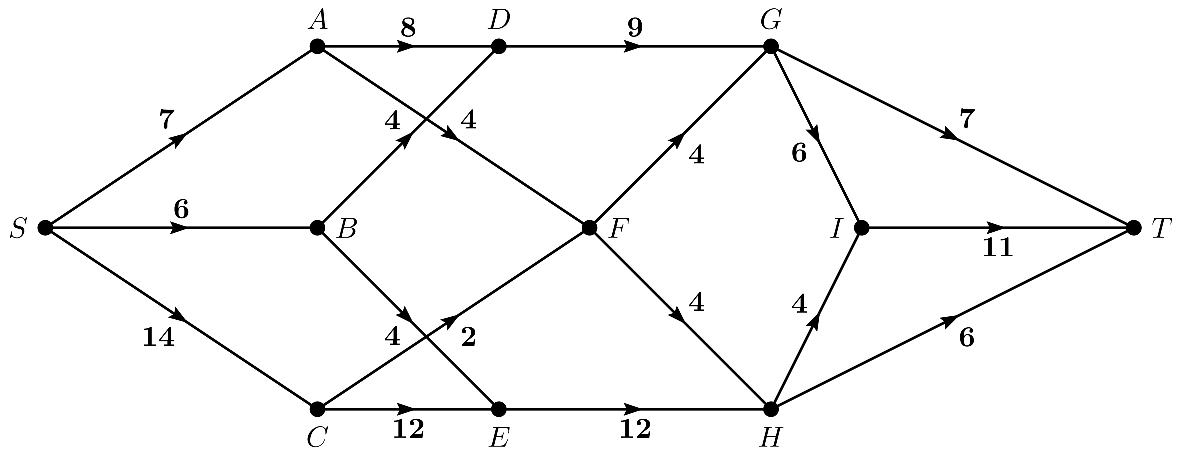
In each case, find:

(a) the smallest number of *links* whose removal would separate the network into parts that could not communicate with each other;

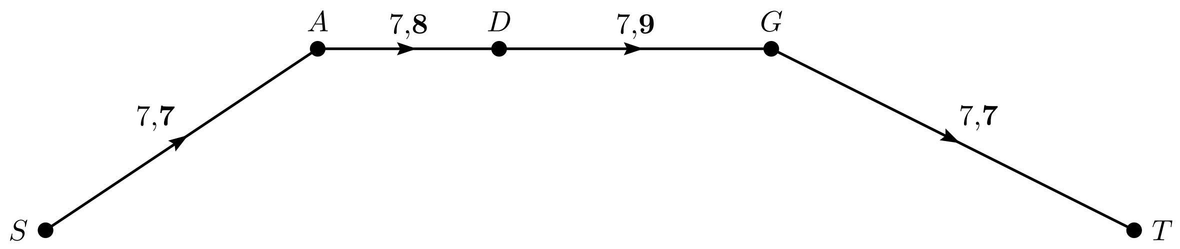
(b) the smallest number of *exchanges* whose removal would separate the *remaining* exchanges into two parts that could not communicate with each other.

Question 2.3 – Network flows

The following diagram represents a network of pipelines along which a fluid (for example, oil or water) flows from a starting point S to a terminal T. Each of the intermediate points A to I represents a pipe junction at which the total flow into the junction must equal the total flow out (so that no fluid is 'lost' on the way). Each line between two junctions represents a pipeline, and the number next to it is the *capacity* of that pipeline (in some units of volume per unit time); the flow along a pipeline must not exceed its capacity, and must be in the direction indicated.



Inspection of the above diagram shows that a *flow* of at most 7 units can be sent along the route $SADGT$ without exceeding the capacity of any of the pipelines SA, AD, DG or GT. This is illustrated in the following diagram, where the first number on each line represents the flow along that pipeline and the second number – in bold type – its capacity.



(a) How can 13 units of fluid per unit time be sent from S to T without exceeding the capacity of any pipeline?

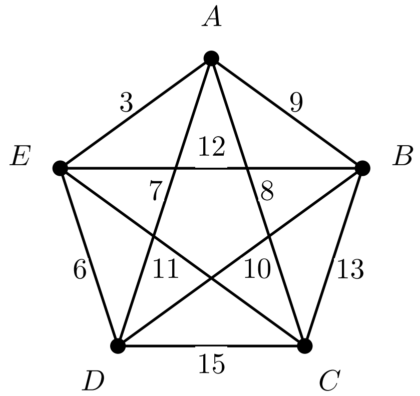
(b) How can 15 units per unit time be sent?

(c) Explain why it is impossible to send more than 23 units per unit time from S to T.

Hint: look at the pipelines DG, FG, HI and HT.

Question 2.4 – Minimum connector problems

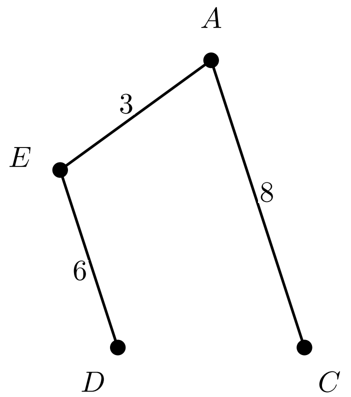
Consider the case of an electricity company that wants to lay a network of cables in order to link together five towns, A, B, C, D and E. It wants to minimise the amount of cabling, in order to keep its costs down. The distances (in miles) between the towns are shown in the following diagram:



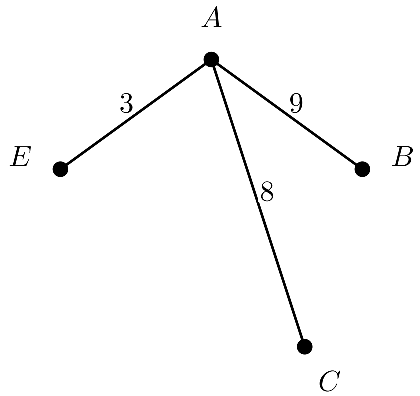
(For example, the distance between A and B is 9 miles and the distance between A and C is 8 miles.)

The company's problem is one of finding a *minimum connector* — a set of links of minimum *total* length that connect all five towns.

For example, a minimum connector that links the towns A, C, D and E (but not B) comprises the links AC, AE and DE. This minimum connector has total length 17 miles.



Similarly, a minimum connector that links the towns A, B, C and E (but not D) comprises the links AB, AC and AE, of total length 20 miles.



(a) Find a minimum connector that links the towns B, C, D and E.

(b) Find a minimum connector that links all five towns.

Question 2.5 – Travelling salesman problems

A travelling salesman wishes to visit a number of towns and return to his starting point, selling his wares as he goes. He wants to select the route with the least total length. Which route should he choose, and how long is it?

Although this type of problem sounds very like a minimum connector problem, it is actually much more difficult to calculate the best possible solution efficiently if there are a large number of cities.

However, solving the minimum connector problem for the same set of cities, *with one removed*, can give a *lower bound* for the solution to the problem, and this question illustrates this rather subtle idea.

Look at the distances shown between towns A, B, C, D and E for the minimum connector problem above. Try to explain why the fact that the minimum connector for towns A, C, D and E has total length 17 miles shows that any solution to the travelling salesman problem for *all five* towns must have total length *at least* 36 miles.

Question 2.6 – Job assignment

A building contractor advertises five jobs — those of bricklayer (b), carpenter (c), decorator (d), electrician (e) and plumber (p).

There are four applicants — one for carpenter and decorator, one for bricklayer, carpenter and plumber, one for decorator, electrician and plumber, and one for carpenter and electrician. Since each job is a full-time post, this means that not all the jobs can be filled. But is it possible for all four applicants to be assigned each to one job for which they are qualified?

In order to solve this problem, it is convenient to represent the information in tabular form, as shown below.

Applicant	Job
1	c, d
2	b, c, p
3	d, e, p
4	c, e

From the table, we can see that one possible assignment of applicants to jobs is:

1 — carpenter
2 — plumber
3 — decorator
4 — electrician

(a) Can you find another assignment of applicants to jobs, so that applicant 2 is assigned the job of carpenter?

(b) List as many other solutions as you can.

Question 2.7 – Pick-up-bricks

Two MST368 students – Rose and Colin – are playing a simple game called Pick-up-bricks. The rules of the game are as follows.

> The game starts with a pile of n bricks. The players take it in turns to remove either 1 or 2 bricks at a time from the pile. Rose goes first. The winner is the player who picks up the last brick.

For each of the following values of n, determine whether one of the two players can play in such a way as to always win, irrespective of how the other player moves.

(a) $n = 3$ (b) $n = 4$ (c) $n = 5$

Question 2.8 – Latin squares

A *latin square* is an arrangement of a set of symbols in a square array such that each symbol appears exactly once in each row and exactly once in each column.

Most people have encountered latin squares in the form of Sudoku puzzles. Latin squares are, however, one of the oldest known algebraic structures, appearing on artefacts from the first century CE. These arrangements of symbols have wide-ranging applications, including in statistics, cryptography and, of course, recreational mathematics.

A *partial latin square* is an arrangement of a set of symbols such that each symbol appears at most once in each row and at most once in each column, for example, the following is an example of a partial latin square with symbol set 1 to 5.

$$P = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & & \\ \hline & & & 5 & \\ \hline 3 & & 5 & & \\ \hline & & 4 & & \\ \hline & 3 & & & \\ \hline \end{array}$$

(a) Fill in the symbols 1 to 5 to complete P to a latin square.

(b) Can you add just one symbol (from $\{1, 2, 3, 4, 5\}$) to P to create another partial latin square that cannot be completed to a latin square?

Solutions

Solution 1.1

(a) $2^6 \times 2^{-3} = 2^{6-3} = 2^3 = 8$.

(b) $2^n \div 2^3 = 2^{n-3}$.

(c) $k^4 \times k^7 \times (k^2)^{-1} = k^4 \times k^7 \times k^{-2} = k^{4+7-2} = k^9$.

Solution 1.2

(a) This is equivalent to

$$2x + 4 \leq 5x + 1$$

which simplifies to

$$3 \leq 3x \text{ or } x \geq 1.$$

(b) This is equivalent to

$$\frac{1}{x} < \frac{1}{4}.$$

Assuming that x is positive, this simplifies to

$$4 < x \text{ or } x > 4.$$

(The simplification step involves multiplying each side by $4x$, and this operation preserves the inequality sign only if $4x$ is positive. This is why we asked you to assume x positive.)

Solution 1.3

(a) $\mathbf{AB} = \begin{bmatrix} 11 & -4 & 2 \\ 25 & -14 & 10 \end{bmatrix}.$

(b) In a matrix product, the rows of the first matrix are combined with the columns of the second to produce the entries of the product. Thus the number of *columns* of the first matrix (which is the number of entries in each *row*) must equal the number of *rows* of the second matrix (which is the number of entries in each *column*). But $\mathbf{B}$ has three columns while $\mathbf{A}$ has two rows.

Solution 1.4

Using arithmetic modulo 2, we have:

(a) $$\mathbf{C}\mathbf{C}_1 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+0+1+0+1 \\ 1+0+1+0+0 \\ 1+0+1+0+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

(b) $$\mathbf{C}\mathbf{C}_2 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+0+1+0+0 \\ 1+0+1+0+0 \\ 1+0+1+0+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solution 1.5

(a) $(0, 1, 0)$, $(1, 0, 0)$, $(1, 0, 1)$, $(1, 1, 1)$.

(b) The cube has twelve edges.

Solution 1.6

The equation $bk = vr$, along with the values of b, v and k gives $r = 3$. Thus,

$$\lambda = \frac{r(k-1)}{v-1} = \frac{3 \times 2}{6} = 1.$$

Solution 1.7

(a) Since ℓ passes through $(-4, 2)$ and $(2, 5)$, we see that it has gradient

$$m = \frac{5-2}{2-(-4)} = \frac{1}{2}.$$

Thus $y = \frac{1}{2}x + c$, and we can substitute $x = 2$, $y = 5$ into this equation to find c:

$$c = y - \tfrac{1}{2}x = 5 - \tfrac{1}{2} \times 2 = 4.$$

Therefore, the equation for ℓ is $y = \frac{1}{2}x + 4$.

(b) If we substitute $x = -1$ into the equation for ℓ, then we obtain:

$$y = \tfrac{1}{2} \cdot (-1) + 4 = \tfrac{7}{2}.$$

Thus, $x = -1$, $y = 3$ is not a solution to the equation for ℓ, and hence we conclude that $(-1, 3)$ does *not* lie on ℓ.

Solution 1.8

$$\begin{aligned}
u_3 &= 4 + (1) = 5 \\
u_4 &= 10 + (2 + 2) = 14 \\
u_5 &= 28 + (5 + 4 + 5) = 42 \\
u_6 &= 84 + (14 + 10 + 10 + 14) = 132.
\end{aligned}$$

Solution 1.9

There are ten triangles: *abc*, *abd*, *abe*, *acd*, *ace*, *ade*, *bcd*, *bce*, *bde*, *cde*.

If there were n dots, there would be n ways of choosing the first corner, then $(n-1)$ ways of choosing the second and finally $(n-2)$ ways of choosing the third.

However, each of these $n(n-1)(n-2)$ sequences of choices describes a triangle with its corners given *in a particular order*, and the three corners of any triangle can be listed in six different orders. (For example, the first triangle of the ten listed above can be given as *abc*, *acb*, *bac*, *bca*, *cab* or *cba*.) Thus the answer is $\dfrac{n(n-1)(n-2)}{6}$.

(If you have met binomial symbols before, you will recognise this as $\dbinom{n}{3}$.)

Solution 1.10

Let P and Q be the following statements:

P means: The student has successfully completed MST124.
Q means: The student is in a good position to study MST368.

You are told that $P \Rightarrow Q$ is true.

(a) You know that P and $P \Rightarrow Q$ are true, and the conclusion is that Q is true. This is a valid deduction. Ayesha has successfully completed MST124 and so is in a good position to study MST368.

(b) You know that Q and $P \Rightarrow Q$ are true, and the conclusion is that P is true. This is not a valid deduction. Beata may be in a good position to study MST368 without first studying MST124 – she may have acquired her mathematical skills by some other means.

(c) You know that P is false, and that $P \Rightarrow Q$ is true, and the conclusion is that Q is false. This is a not a valid deduction. Connor may not have studied MST124 but he may be in a good position to study MST368 having acquired his mathematical skills by some other means.

Solution 1.11

There are several valid ways to prove the statement; here we give three.

Proof by induction

Let $P(n)$ be the statement $1 + 3 + 5 + \cdots + (2n - 1) = n^2$. Then $P(1)$ is true because $1 = 1^2$.

Now, let $k \geq 1$, and assume $P(k)$ is true; that is, $1+3+\cdots+(2k-1) = k^2$.

We want to deduce that $P(k + 1)$ is true; that is,
$1 + 3 + \cdots + (2k + 1) = (k + 1)^2$.

Now

$$\begin{aligned} 1 + 3 + \cdots + (2k + 1) &= 1 + 3 + \cdots + (2k - 1) + (2k + 1) \\ &= k^2 + (2k + 1) \quad \text{by } P(k) \\ &= (k + 1)^2. \end{aligned}$$

That is, $P(k + 1)$ is true.

Thus, $P(k + 1)$ is true if $P(k)$ is true; that is, $P(k) \Rightarrow P(k + 1)$, for all $k = 1, 2, \ldots$.

Hence by mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

A direct proof

We can pair up the terms on the left-hand side to make $2n$:
$1 + (2n - 1) = 2n$; $3 + (2n - 3) = 2n$, and so on.

If n is even, then there are $n/2$ pairs of terms, each summing to $2n$:

$$\begin{array}{cccc} 1 & 3 & \cdots & n-1 \\ 2n-1 & 2n-3 & \cdots & n+1 \end{array}$$

So, the total is $\frac{n}{2}2n = n^2$.

If n is odd, then $n - 2$ and $n + 2$ are a pair, but n is not in any pair. This gives n, plus $(n - 1)/2$ pairs of terms, each summing to $2n$:

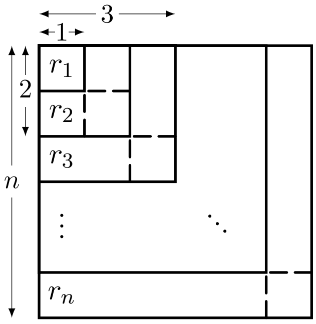
$$\begin{array}{ccccc} 1 & 3 & \cdots & n-1 & n \\ 2n-1 & 2n-3 & \cdots & n+1 & \end{array}$$

So, the total is $n + \frac{(n-1)}{2}2n = n + n(n-1) = n^2$.

In both cases, $1 + 3 + 5 + \cdots + (2n-1) = n^2$ for all $n \in \mathbb{N}$, as required.

A direct proof using pictures

Consider a square S_n of side n, where $n \in \mathbb{N}$. Then the area of S_n is n^2 (units^2). We can divide S_n into n regions r_1 to r_n as follows:



We observe:

The area of region r_1 is 1 (unit^2)
The area of region r_2 is $1 + 1 + 1 = 3$ (units^2)
The area of region r_3 is $2 + 2 + 1 = 5$ (units^2)

and so on up to:

The area of region r_n is $(n-1) + (n-1) + 1 = 2n - 1$ (units^2).

Clearly, the areas of the regions sum to the area of the square S_n, so

$$1 + 3 + 5 + \cdots + (2n-1) = n^2, \text{ for all } n \in \mathbb{N}.$$

Solution 2.1

No, the map cannot be coloured with just three colours. Three colours are needed for the ring of five states surrounding Nevada (since if we try to colour them with two colours alternating, we find that two adjacent states in the ring must have the same colour). Now Nevada is adjacent to all of these states, and hence requires a fourth colour.

Although it may be difficult to see, actually the whole map can be coloured with just four colours.

Solution 2.2

(a) For network (1), the smallest number of links whose closure would separate the network is 2. The closure of any of the pairs: AB and AF; BC and FE; or CD and ED; would separate the network.

For network (2), the smallest number of links whose closure would separate the network is 3. AB, AE and AF; CB, CD and CE; DB, DC and DE; or FA, FB and FE; would separate the network.

(b) For network (1), the smallest number of exchanges whose closures would separate the remaining changes is 2. The closure of any of the pairs: B and E; B and F; C and E; or C and F; would separate the network.

For network (2), the smallest number of exchanges whose closure would separate the remaining exchanges is also 2. The only pair that would do this is: B and E.

Solution 2.3

(a) We can, for example, send 7 units of fluid per unit time along the route $SADGT$ and 6 along the route $SCEHT$.

(b) We can, for example, augment the flow described above, by sending a further 2 units of fluid per unit time along the route $SBDGIT$.

(c) Each route from S to T passes through one of the pipelines DG, FG, HI and HT. Therefore, if we imagine drawing a line across these four pipelines, no more than $9 + 4 + 4 + 6 = 23$ units of fluid can cross that line per unit time.

Solution 2.4

(a) The minimum connector that links the towns B, C, D and E comprises the links BD, CE and DE, of total length 27 miles.

(b) The minimum connector that links all five towns comprises the links AE, ED, AC and AB, of total length 26 miles. (Yes, linking all five towns can be done in a shorter length than linking just the four towns B, C, D and E!)

Solution 2.5

Any solution involves a round trip, and so can start and finish at any town we choose. Let us decide to start and finish it at B. Thus, the salesman must proceed from B to one of the other towns; *then* must cover the other four towns; *then* must return to B along a *different* road than that from which he left B. The first and last of these stages must be at least as long as the sum of the *shortest two* routes from B ($9 + 10 = 19$ miles), while the middle stage must be *at least as long* as a minimum connector for the towns other than B; which we have seen to comprise 17 miles. Thus, any solution for the five towns must have total length at least
$19 + 17 = 36$ miles.

Solution 2.6

(a) Yes, it is possible, as the following assignment can be made:

1: decorator; 2: carpenter; 3: plumber; 4: electrician

(b) The complete list of solutions is as follows.

1: carpenter;	2: bricklayer;	3: plumber;	4: electrician
1: carpenter;	2: bricklayer;	3: decorator;	4: electrician
1: carpenter;	2: plumber;	3: decorator;	4: electrician
1: decorator;	2: bricklayer;	3: electrician;	4: carpenter
1: decorator;	2: bricklayer;	3: plumber;	4: carpenter
1: decorator;	2: bricklayer;	3: plumber;	4: electrician
1: decorator;	2: carpenter;	3: plumber;	4: electrician
1: decorator;	2: plumber;	3: electrician;	4: carpenter.

Solution 2.7

(a) When there are $n = 3$ bricks, Colin (as the second player) can always win. If Rose picks up 1 brick, there are 2 left in the pile, while if Rose picks up 2 bricks, there is 1 brick left. In either case, Colin can pick up all the remaining bricks in the pile and win.

(b) When $n = 4$, Rose can always win. Her first move should be to pick up 1 brick, which leaves 3 bricks in the pile. Now regardless whether Colin picks up 1 or 2 bricks, Rose wins by picking up all the remaining bricks.

(c) When $n = 5$, Rose can always win. Her first move should be to pick up 2 bricks. This leaves 3 bricks in the pile for Colin, and the same argument as in part (b) now applies.

Note: There is a general strategy: on each turn, each player should (if possible) pick up bricks so that the number of bricks left in the pile is a multiple of 3.

This means that Colin can always win if the starting position contains $n = 3k$ bricks for some k, and Rose can win if $n = 3k + 1$ or $n = 3k + 2$.

Solution 2.8

(a) The partial latin square P completes (uniquely) to the following latin square.

1	2	3	4	5
4	1	2	5	3
3	4	5	1	2
2	5	4	3	1
5	3	1	2	4

To see this, you might first notice that two entries in the top row are forced, and then there is only one way to insert '5' into the two bottom rows.

(b) Since the completion of P to a latin square was unique, we can construct another partial latin square from P that *can't* be completed by inserting a number into a currently empty cell so that: (1) that number is not the number needed for the completion found in part (a), and (2) the resulting array is still a partial latin square (that is, no row or column contains any number more than once). For example:

1	2	3		4
			5	
3		5		
		4		
	3			