MST124

Essential mathematics 1

Unit 5

Coordinate geometry and vectors

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5 Vectors

This section marks the start of the second part of this unit, in which you'll learn about a type of mathematical object called a *vector*. Vectors play an important role in the study and analysis of phenomena in physics and engineering.

5.1 What is a vector?

Some mathematical quantities can't be specified just by stating their size – instead you need to state a size *and a direction*. For example, to fully describe the motion of a ship on the ocean at a point in time during its voyage, it's not enough to specify how fast the ship is moving – you also need to describe its direction of motion.

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You can move only in one dimension along a tightrope (unless you fall off!)

You saw other examples of this in Unit 2, when you considered objects moving along straight lines. To specify the position of a point P on a straight line relative to some other point, say O, on the line, you first choose one direction along the line to be the positive direction; then you state the distance between the two points, and attach a plus or minus sign to indicate the direction. The resulting quantity is called the *displacement* of P from O. For example, in Figure 28, if the positive direction is taken to be to the right, then the displacement of A from O is $-2\,\text{cm}$, and the displacement of B from O is $4\,\text{cm}$.

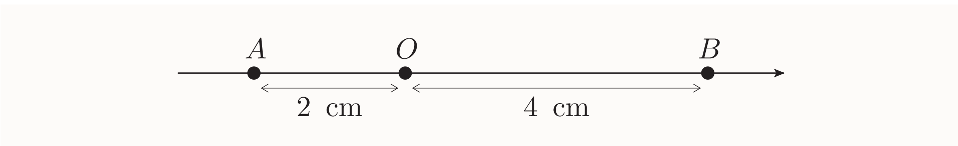

Figure 28 Positions along a straight line

You can move in two dimensions across a flat field

Similarly, if an object is moving along a straight line, then you can describe its motion by giving its speed, and attaching a plus or minus sign to indicate its direction. The resulting quantity is called the *velocity* of the object.

Plus and minus signs provide a convenient way to specify direction when you're dealing with movement along a straight line – that is, in one dimension. Examples of movement of this type include the motion of a car along a straight road, or that of a tightrope walker along a tightrope. However, we often need to deal with movement in two or three dimensions. For example, someone standing in a flat field can move across the field in two dimensions, and a person in space can move in three dimensions.

You can move in three dimensions in space

In general, **displacement** is the position of one point relative to another, whether in one, two or three dimensions. To specify a displacement, you need to give both a distance and a direction. For example, consider the points O, P and Q in Figure 29. You can specify the displacement of P

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from O by saying that it is 1 km north-west of O. Similarly, you can specify the displacement of Q from O by saying that it is 1 km north-east of O.

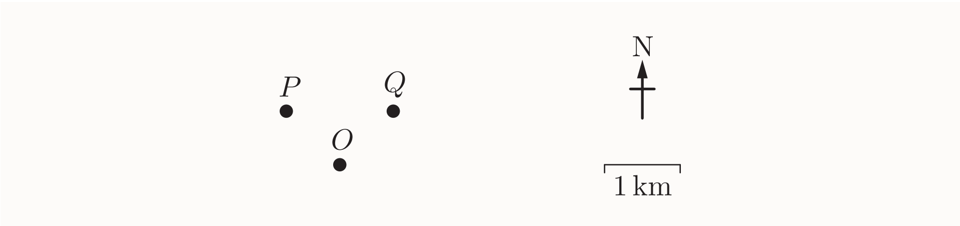


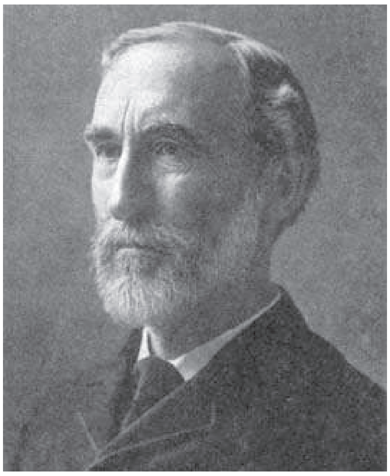
Figure 29 Points in a plane

Just as distance together with direction is called *displacement*, so speed together with direction is called **velocity**. For example, if you say that someone is walking at a speed of $5\,\mathrm{km\,h^{-1}}$ south, then you're specifying a velocity.

Quantities, such as displacement and velocity, that have both a size and a direction are called **vectors**, or **vector quantities**. (In Latin, the word *vector* means 'carrier'.) Another example of a vector quantity is *force*. The size of a vector is usually called its **magnitude**.

In contrast to vectors, quantities that have size but no direction are called **scalars**, or **scalar quantities**. Examples of scalars include distance, speed, time, temperature and volume. So a scalar is a number, possibly with a unit.

Notice that the magnitude of the displacement of one point from another is the distance between the two points, and the magnitude of the velocity of an object is its speed. In everyday English the words 'speed' and 'velocity' are often used interchangeably, but in scientific and mathematical terminology there is an important difference: speed is a scalar and velocity is a vector.



Josiah Willard Gibbs (1839–1903)

The concept of vectors evolved over a long time. Isaac Newton (1642–1727) dealt extensively with vector quantities, but never formalised them. The first exposition of what we would today know as vectors was by Josiah Willard Gibbs in 1881, in his *Elements of vector analysis*. This work was derived from earlier ideas of William Rowan Hamilton (1805–1865).

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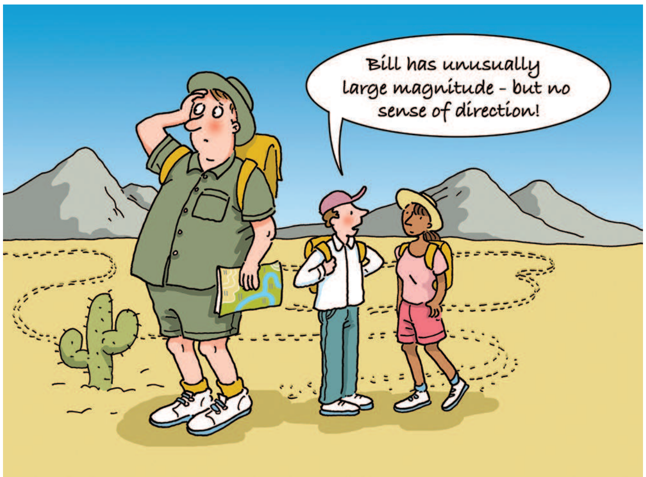

Any vector with non-zero magnitude can be represented by an *arrow*, which is a line segment with an associated direction, like the one in Figure 30. The length of the arrow represents the magnitude of the vector, according to some chosen scale, and the direction of the arrow represents the direction of the vector. Two-dimensional vectors are represented by arrows in a plane, and three-dimensional vectors are represented by arrows in three-dimensional space. For example, the arrow in Figure 30 might represent a displacement of 30 km north-west, if you're using a scale of 1 cm to represent 10 km. Alternatively, the same arrow might represent a velocity of $30\,\mathrm{m\,s^{-1}}$ north-west, if you're using a scale of 1 cm to represent $10\,\mathrm{m\,s^{-1}}$.

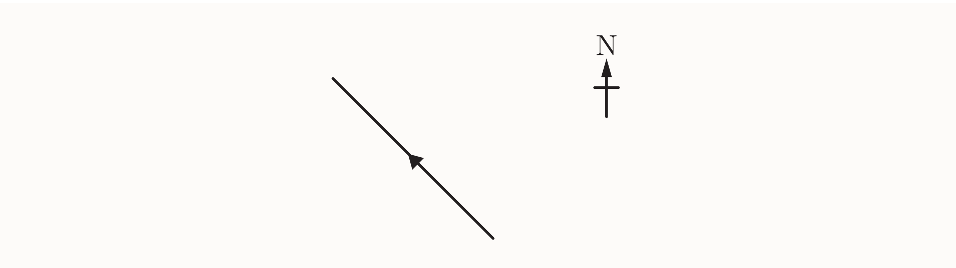

Figure 30 An arrow that represents a vector

Once you've chosen a scale, any two arrows with the same length and the same direction represent the *same* vector. For example, all the arrows in Figure 31 represent the same vector.

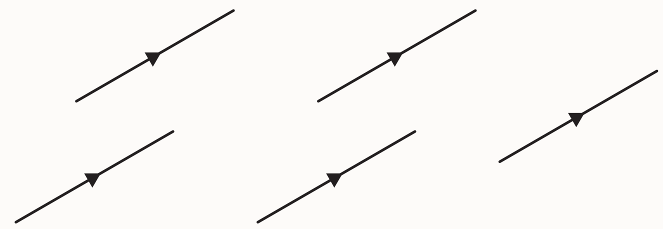

Figure 31 Several arrows representing the same vector

Vectors are often denoted by lower-case letters. We distinguish them from scalars by using a bold typeface in typed text, and by underlining them in handwritten text. For example, the vector in Figure 30 might be denoted by $\mathbf{v}$ in print, or handwritten as $\underline{v}$. These conventions prevent readers from confusing vector and scalar quantities.

> Remember to underline handwritten vectors (and make typed ones bold) in your own work.

Vectors that represent displacements are sometimes called **displacement vectors**. There is a useful alternative notation for such vectors. If P and Q are any two points, then the vector that specifies the displacement *from* P *to* Q (illustrated in Figure 32) is denoted by $\overrightarrow{PQ}$.

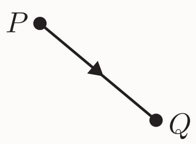

Figure 32 The vector $\overrightarrow{PQ}$

The magnitude of a vector $\mathbf{v}$ is a scalar quantity. It is denoted by $|\mathbf{v}|$, which is read as 'the magnitude of v', 'the modulus of v', or simply 'mod v'. For example, if the vector $\mathbf{v}$ represents a velocity of $30\,\mathrm{m\,s^{-1}}$ north-west, then $|\mathbf{v}| = 30\,\mathrm{m\,s^{-1}}$. Similarly, if the vector $\overrightarrow{PQ}$ represents a displacement of $3\,\mathrm{m}$ south-east, then $|\overrightarrow{PQ}| = 3\,\mathrm{m}$. Remember that the distance between the points P and Q can also be denoted by PQ, so

$$PQ = |\overrightarrow{PQ}|.$$

Notice that the notation for the magnitude of a vector is the same as the notation for the magnitude of a scalar that you met in Unit 3. For example, you saw there that $|-3| = 3$. So this notation can be applied to either vectors or scalars.

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When we're working with vectors, it's often convenient, for simplicity, not to distinguish between vectors and the arrows that represent them. For example, we might say 'the vector shown in the diagram' rather than 'the vector represented by the arrow shown in the diagram'. This convention is used throughout the rest of this unit.

Over the next few pages you'll learn the basics of working with vectors.

Equal vectors

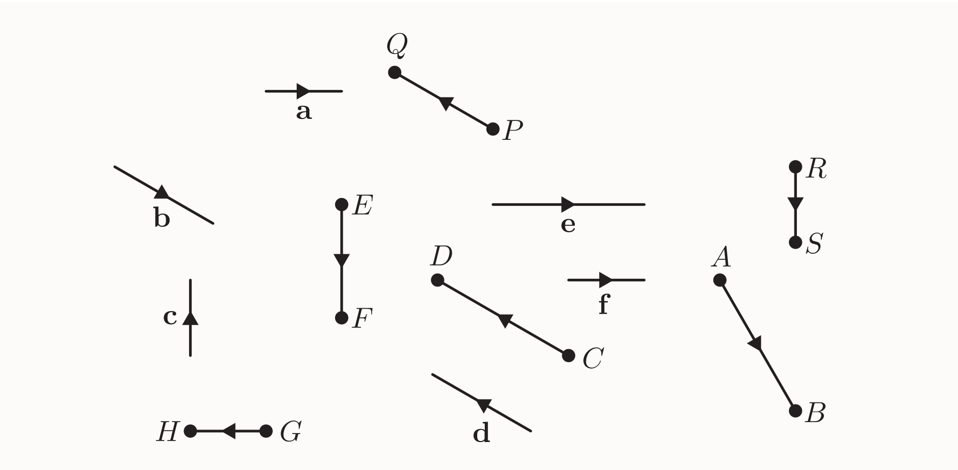
As you'd expect, two vectors are **equal** if they have the same magnitude and the same direction.

Activity 23 *Identifying equal vectors*

The following diagram shows several displacement vectors.

(a) Which vector is equal to the vector $\mathbf{a}$?

(b) Which vector is equal to the vector $\overrightarrow{PQ}$?



The zero vector

The *zero vector* is defined as follows.

Zero vector

The **zero vector**, denoted by $\mathbf{0}$ (bold zero), is the vector whose magnitude is zero. It has no direction.

The zero vector is handwritten as $\underline{0}$ (zero underlined).

For example, the displacement of a particular point from itself is the zero vector, as is the velocity of an object that is not moving.

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Addition of vectors

To understand how to *add* two vectors, it's helpful to think about displacement vectors. For example, consider the situation shown in Figure 33. Suppose that an object is positioned at a point P and you first move it to the point Q, then you move it again to the point R. The two displacements are the vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$, respectively, and the overall, combined displacement is the vector $\overrightarrow{PR}$. This method of combining two vectors to produce another vector is called **vector addition**.

We write

$$\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}.$$

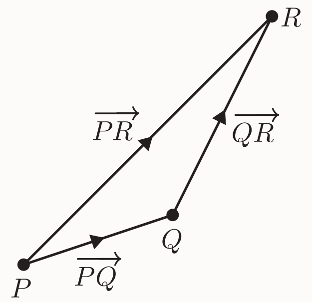
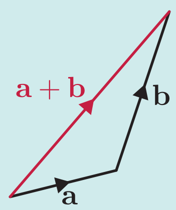


Figure 33 The result of adding two displacement vectors

Vectors are always added in this way. The general rule is called the *triangle law for vector addition*, and it can be stated as follows.

Triangle law for vector addition

To find the sum of two vectors **a** and **b**, place the tail of **b** at the tip of **a**. Then $\mathbf{a} + \mathbf{b}$ is the vector from the tail of **a** to the tip of **b**.



The sum of two vectors is also called their **resultant** or **resultant vector**.

You can add two vectors in either order, and you get the same result either way. This is illustrated in Figure 34. Diagrams (a) and (b) show how the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{b} + \mathbf{a}$ are found using the triangle law for vector addition. When you place these two diagrams together, as shown in diagram (c), the two resultant vectors coincide, because they lie along the diagonal of the parallelogram formed by the two copies of **a** and **b**.

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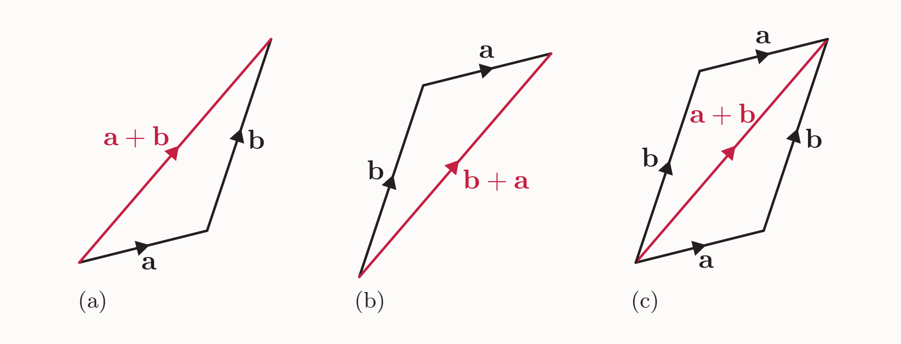
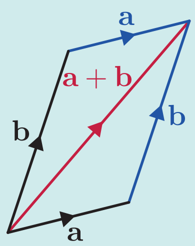


Figure 34 The vectors **a** + **b** and **b** + **a** are equal

In fact, Figure 34(c) gives an alternative way to add two vectors, the *parallelogram law for vector addition*, which can be stated as follows.

Parallelogram law for vector addition

To find the sum of two vectors **a** and **b**, place their tails together, and complete the resulting figure to form a parallelogram. Then **a** + **b** is the vector formed by the diagonal of the parallelogram, starting from the point where the tails of **a** and **b** meet.



The parallelogram law is convenient in some contexts, and you'll use it in Unit 12. In this unit we'll always use the triangle law, as it's simpler in the sorts of situations that we'll deal with here.

You can add more than two vectors together. To add several vectors, you place them all tip to tail, one after another; then their sum is the vector from the tail of the first vector to the tip of the last vector. For example, Figure 35 illustrates how three vectors **a**, **b** and **c** are added.

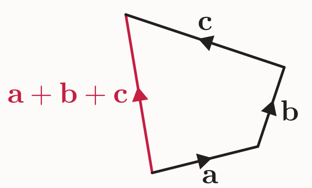


Figure 35 The sum of three vectors **a**, **b** and **c**

The order in which you add the vectors doesn't matter – you always get the same resultant vector.

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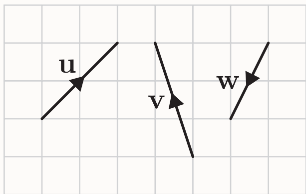
Activity 24 *Adding vectors*

The diagram below shows three vectors **u**, **v** and **w** drawn on a grid.



Draw arrows representing the following vector sums. (Use squared paper or sketch a grid.)

(a) $\mathbf{u} + \mathbf{v}$ (b) $\mathbf{u} + \mathbf{w}$ (c) $\mathbf{v} + \mathbf{w}$ (d) $\mathbf{u} + \mathbf{v} + \mathbf{w}$ (e) $\mathbf{u} + \mathbf{u}$

As you'd expect, adding the zero vector to any vector leaves it unchanged. That is, for any vector **a**,

$$\mathbf{a} + \mathbf{0} = \mathbf{a}.$$

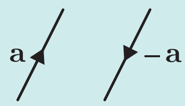
Note that you can't add a vector to a scalar. Expressions such as $\mathbf{v} + x$, where **v** is a vector and x is a scalar, are meaningless.

Negative of a vector

The *negative* of a vector **a** is denoted by $-\mathbf{a}$, and is defined as follows.

Negative of a vector

The **negative** of a vector **a**, denoted by $-\mathbf{a}$, is the vector with the same magnitude as **a**, but the opposite direction.



For any points P and Q, the position vectors $\overrightarrow{PQ}$ and $\overrightarrow{QP}$ have the property that $-\overrightarrow{PQ} = \overrightarrow{QP}$, since $\overrightarrow{PQ}$ and $\overrightarrow{QP}$ have opposite directions.

If you add any vector **a** to its negative $-\mathbf{a}$, by placing the two vectors tip to tail in the usual way, then you get the zero vector. In other words, for any vector **a**,

$$\mathbf{a} + (-\mathbf{a}) = \mathbf{0},$$

as you'd expect.

The negative of the zero vector is the zero vector; that is, $-\mathbf{0} = \mathbf{0}$.

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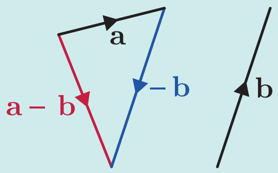
Subtraction of vectors

To see how vector *subtraction* is defined, first consider the subtraction of numbers. Subtracting a number is the same as adding the negative of the number. In other words, if a and b are numbers, then $a - b$ means the same as $a + (-b)$. We use this idea to define vector subtraction, as follows.

Vector subtraction

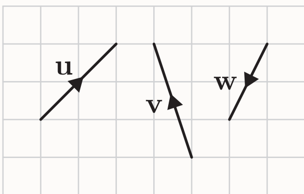
To subtract $\mathbf{b}$ from $\mathbf{a}$, add $-\mathbf{b}$ to $\mathbf{a}$. That is,

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}).$$



Activity 25 *Subtracting vectors*

The diagram below shows three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ drawn on a grid.



Draw arrows representing the following negatives and differences of vectors. (Use squared paper or sketch a grid.)

(a) $-\mathbf{v}$ (b) $-\mathbf{w}$ (c) $\mathbf{u} - \mathbf{v}$ (d) $\mathbf{v} - \mathbf{w}$ (e) $\mathbf{u} + \mathbf{v} - \mathbf{w}$

You have already seen that for any vector $\mathbf{a}$, we have $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$. That is, for any vector $\mathbf{a}$, we have $\mathbf{a} - \mathbf{a} = \mathbf{0}$, as you would expect.

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Multiplication of vectors by scalars

You can multiply vectors by scalars. To understand what this means, first consider the effect of adding a vector $\mathbf{a}$ to itself, as illustrated in Figure 36.

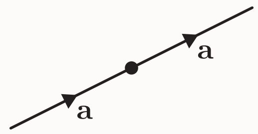

Figure 36 A vector $\mathbf{a}$ added to itself

The resultant vector $\mathbf{a} + \mathbf{a}$ has the same direction as $\mathbf{a}$, but twice the magnitude. We denote it by $2\mathbf{a}$. We say that this vector is a **scalar multiple** of the vector $\mathbf{a}$, since 2 is a scalar quantity. In general, scalar multiplication of vectors is defined as below. Note that in this box the notation $|m|$ means the magnitude of the *scalar* m.

Scalar multiple of a vector

Suppose that $\mathbf{a}$ is a vector. Then, for any non-zero real number m, the **scalar multiple** $m\mathbf{a}$ of $\mathbf{a}$ is the vector

- whose magnitude is $|m|$ times the magnitude of $\mathbf{a}$
- that has the same direction as $\mathbf{a}$ if m is positive, and the opposite direction if m is negative.

Also, $0\mathbf{a} = \mathbf{0}$.
(That is, the number zero times the vector $\mathbf{a}$ is the zero vector.)

Remember that a scalar multiple of a vector is a *vector*.

Various scalar multiples of a vector $\mathbf{a}$ are shown in Figure 37.

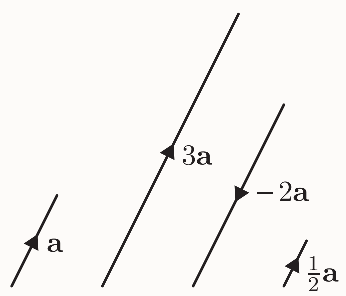

Figure 37 Scalar multiples of a vector $\mathbf{a}$

By the definition above, if $\mathbf{a}$ is any vector, then $(-1)\mathbf{a}$ is the vector with the same magnitude as $\mathbf{a}$ but the opposite direction. In other words, as

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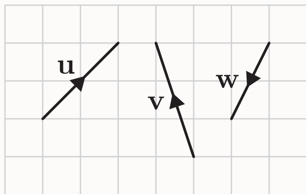
you would expect,

$$(-1)\mathbf{a} = -\mathbf{a}.$$

Activity 26 *Multiplying vectors by scalars*

The diagram below shows three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ drawn on a grid.



Draw arrows representing the following vectors. (Use squared paper or sketch a grid.)

(a) $3\mathbf{u}$ (b) $-2\mathbf{v}$ (c) $\frac{1}{2}\mathbf{v}$ (d) $3\mathbf{u} - 2\mathbf{v}$ (e) $-2\mathbf{v} + \mathbf{w}$

The next example illustrates how you can use scalar multiples of vectors to represent quantities in practical situations.

Example 10 *Scaling velocities*

Suppose that the vector $\mathbf{u}$ represents the velocity of a car travelling with speed $50\,\mathrm{km\,h^{-1}}$ along a straight road heading north. Write down, in terms of $\mathbf{u}$, the velocity of a second car that is travelling in the same direction as the first with a speed of $75\,\mathrm{km\,h^{-1}}$.

Solution

The velocity of the second car has the same direction as $\mathbf{u}$, and hence is a scalar multiple of it. The speed of the second car is 1.5 times the speed of the first.

The velocity of the second car is $1.5\mathbf{u}$.

The activity below involves winds measured in *knots*. A knot is a unit of speed often used in meteorology, and in air and maritime navigation. Its usual abbreviation is kn, and $1\,\mathrm{kn} = 1.852\,\mathrm{km\,h^{-1}}$.

Conventionally, the direction of a wind is usually given as the direction *from* which it blows, rather than the direction that it blows towards. So, for example, a southerly wind is one blowing from the south, towards the north.

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Activity 27 *Scaling velocities*

Suppose that the vector $\mathbf{v}$ represents the velocity of a wind of 35 knots blowing from the north-east. Express the following vectors in terms of $\mathbf{v}$.

(a) The velocity of a wind of 70 knots blowing from the north-east.

(b) The velocity of a wind of 35 knots blowing from the south-west.

5.2 Vector algebra

In Subsection 5.1, you met some properties of the addition, subtraction and scalar multiplication of vectors. For example, you saw that for any vectors $\mathbf{a}$ and $\mathbf{b}$,

$$\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}, \quad \mathbf{a}+\mathbf{0}=\mathbf{a}, \quad \mathbf{a}-\mathbf{a}=\mathbf{0} \quad \text{and} \quad \mathbf{a}+\mathbf{a}=2\mathbf{a}.$$

All the properties that you met can be deduced from the eight basic algebraic properties of vectors listed below.

Properties of vector algebra

The following properties hold for all vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, and all scalars m and n.

1. $\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$
2. $(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c})$
3. $\mathbf{a}+\mathbf{0}=\mathbf{a}$
4. $\mathbf{a}+(-\mathbf{a})=\mathbf{0}$
5. $m(\mathbf{a}+\mathbf{b})=m\mathbf{a}+m\mathbf{b}$
6. $(m+n)\mathbf{a}=m\mathbf{a}+n\mathbf{a}$
7. $m(n\mathbf{a})=(mn)\mathbf{a}$
8. $1\mathbf{a}=\mathbf{a}$

These properties are similar to properties that hold for addition, subtraction and multiplication of real numbers. Similar properties also hold for many different systems of mathematical objects. You'll meet further examples of such systems later in the module.

Property 1 says that the order in which two vectors are added does not matter. This property can be described by saying that vector addition is **commutative**. Similarly, addition of real numbers is commutative, because $a+b=b+a$ for all real numbers a and b, and multiplication of real numbers is commutative, because $ab=ba$ for all real numbers a and b. Subtraction of real numbers is not commutative, because it is not true that $a-b=b-a$ for all real numbers a and b.

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Property 2 says that finding $\mathbf{a} + \mathbf{b}$ and then adding $\mathbf{c}$ to the result gives the same final answer as finding $\mathbf{b} + \mathbf{c}$ and then adding $\mathbf{a}$ to the result. You might like to check this for a particular case, by drawing the vectors as arrows. This property is described by saying that vector addition is **associative**. It allows us to write the expression $\mathbf{a} + \mathbf{b} + \mathbf{c}$ without there being any ambiguity in what is meant – you can interpret it as either $(\mathbf{a} + \mathbf{b}) + \mathbf{c}$ or $\mathbf{a} + (\mathbf{b} + \mathbf{c})$, because both mean the same. Addition and multiplication of real numbers are also associative operations.

Property 5 says that adding two vectors and then multiplying the result by a scalar gives the same final answer as multiplying each of the two vectors individually by the scalar and then adding the two resulting vectors. This property is described by saying that scalar multiplication is **distributive** over the addition of vectors.

Similarly, property 6 says that scalar multiplication is distributive over the addition of scalars.

You will notice that nothing has been said about whether vectors can be multiplied or divided by other vectors. There is a useful way to define multiplication of two vectors – two different ways, in fact! You will meet one of these ways in Section 7. Division of a vector by another vector is not possible.

The properties in the box above allow you to perform some operations on vector expressions in a similar way to real numbers, as illustrated in the following example.

Example 11 *Simplifying a vector expression*

Simplify the vector expression

$$2(\mathbf{a} + \mathbf{b}) + 3(\mathbf{b} + \mathbf{c}) - 5(\mathbf{a} + \mathbf{b} - \mathbf{c}).$$

Solution

Expand the brackets, using property 5 above.

$$\begin{aligned} &2(\mathbf{a} + \mathbf{b}) + 3(\mathbf{b} + \mathbf{c}) - 5(\mathbf{a} + \mathbf{b} - \mathbf{c}) \\ &= 2\mathbf{a} + 2\mathbf{b} + 3\mathbf{b} + 3\mathbf{c} - 5\mathbf{a} - 5\mathbf{b} + 5\mathbf{c} \end{aligned}$$

Collect like terms, using property 6 above.

$$\begin{aligned} &= 2\mathbf{a} - 5\mathbf{a} + 2\mathbf{b} + 3\mathbf{b} - 5\mathbf{b} + 3\mathbf{c} + 5\mathbf{c} \\ &= 8\mathbf{c} - 3\mathbf{a}. \end{aligned}$$

The properties in the box above also allow you to manipulate equations containing vectors, which are known as **vector equations**, in a similar way to ordinary equations. For example, you can add or subtract vectors on both sides of such an equation, and you can multiply or divide both

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sides by a non-zero scalar. You can use these methods to rearrange a vector equation to make a particular vector the subject, or to solve a vector equation for an unknown vector.

Activity 28 *Manipulating vector expressions and equations*

(a) Simplify the vector expression $4(\mathbf{a}-\mathbf{c})+3(\mathbf{c}-\mathbf{b})+2(2\mathbf{a}-\mathbf{b}-3\mathbf{c})$.

(b) Rearrange each of the following vector equations to express $\mathbf{x}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(i) $2\mathbf{b}+4\mathbf{x}=7\mathbf{a}$ (ii) $3(\mathbf{b}-\mathbf{a})+5\mathbf{x}=2(\mathbf{a}-\mathbf{b})$

5.3 Using vectors

In this subsection you'll see some examples of how you can use two-dimensional vectors in practical situations.

When you use a vector to represent a real-world quantity, you need a means of expressing its direction. For a two-dimensional vector, one way to do this is to state the angle measured from some chosen reference direction to the direction of the vector. You have to make it clear whether the angle is measured clockwise or anticlockwise.

A navigational compass

If the vector represents the displacement or velocity of an object such as a ship or an aircraft, then its direction is often given as a *compass bearing*. There are various different types of compass bearings, but in this module we will use the following type.

A **bearing** is an angle between 0° and 360°, measured clockwise in degrees from north to the direction of interest.

For example, Figure 38 shows a vector $\mathbf{v}$ with a bearing of 150°.

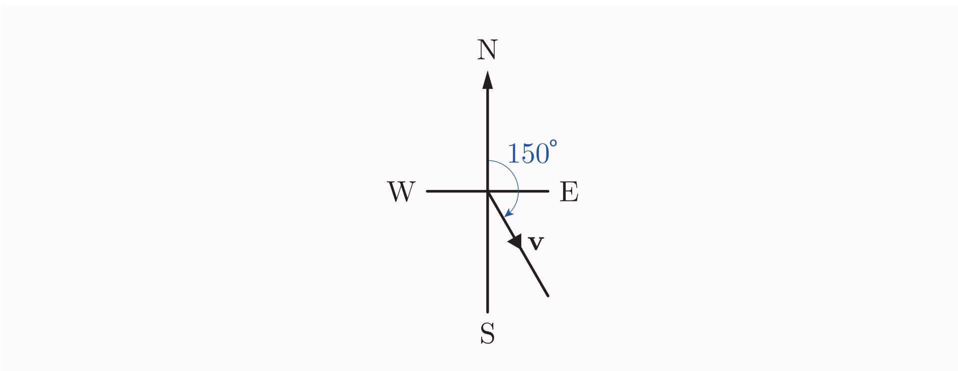

Figure 38 A vector with a bearing of 150°

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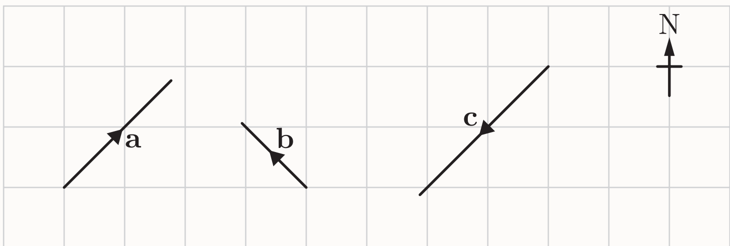
When bearings are used in practice, there are various possibilities for the meaning of 'north'. It can mean magnetic north (the direction in which a compass points), true north (the direction to the North Pole) or grid north (the direction marked as north on a particular map). We'll assume that one of these has been chosen in any particular situation.

Notice that the rotational direction in which bearings are measured is *opposite* to that in which angles are usually measured in mathematics.

Bearings are measured *clockwise* (from north), whereas in Unit 4 you saw angles measured *anticlockwise* (from the positive direction of the x-axis).

Activity 29 *Working with bearings*

(a) Write down the bearings that specify the directions of the following vectors. (The acute angle between each vector and the gridlines is 45°.)



(b) Draw arrows to represent vectors (of any magnitude) with directions given by the following bearings.

(i) 90° (ii) 135° (iii) 270°

When you work with the directions of vectors expressed using angles, you often need to use trigonometry, as illustrated in the next example.

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Example 12 *Adding two perpendicular vectors*

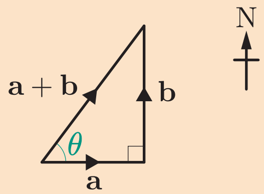
An explorer walks for 3 km on a bearing of 90°, then turns and walks for 4 km on a bearing of 0°.

Find the magnitude and bearing of his resultant displacement, giving the bearing to the nearest degree.

Solution

Represent the first part of the walk by the vector $\mathbf{a}$, and the second part by the vector $\mathbf{b}$. Then the resultant displacement is $\mathbf{a}+\mathbf{b}$.

Draw a diagram showing $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a}+\mathbf{b}$. Since $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, you obtain a right-angled triangle.



Use Pythagoras' theorem to find the magnitude of $\mathbf{a}+\mathbf{b}$.

Since $|\mathbf{a}| = 3\,\text{km}$, $|\mathbf{b}| = 4\,\text{km}$ and the triangle is right-angled,

$$|\mathbf{a}+\mathbf{b}| = \sqrt{|\mathbf{a}|^2 + |\mathbf{b}|^2} = \sqrt{3^2+4^2} = \sqrt{25} = 5\,\text{km}.$$

Use basic trigonometry to find one of the acute angles in the triangle.

From the diagram,

$$\tan\theta = \frac{|\mathbf{b}|}{|\mathbf{a}|} = \frac{4}{3},$$

so $\theta = \tan^{-1}\left(\frac{4}{3}\right) = 53°$ (to the nearest degree).

Hence find the bearing of $\mathbf{a}+\mathbf{b}$.

The bearing of $\mathbf{a}+\mathbf{b}$ is $90° - \theta = 37°$ (to the nearest degree).

State a conclusion, remembering to include units.

So the resultant displacement has magnitude 5 km and a bearing of approximately 37°.

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Activity 30 *Adding two perpendicular vectors*

A yacht sails on a bearing of $60°$ for 5.3 km, then turns through $90°$ and sails on a bearing of $150°$ for a further 2.1 km.

Find the magnitude and bearing of the yacht's resultant displacement. Give the magnitude of the displacement in km to one decimal place, and the bearing to the nearest degree.

The vectors that were added in Example 12 and Activity 30 were perpendicular, so only basic trigonometry was needed. In the next activity, you're asked to add two displacement vectors that aren't perpendicular. You need to draw a clear diagram and use the sine and cosine rules to find the required lengths and angles.

Activity 31 *Adding two non-perpendicular vectors*

The grab of a robotic arm moves 40 cm from its starting point on a bearing of $90°$ to pick up an object, and then moves the object 20 cm on a bearing of $315°$.

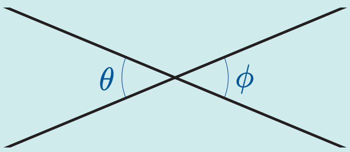
Find the resultant displacement of the grab, giving the magnitude to the nearest centimetre, and the bearing to the nearest degree.

In some examples involving vectors, it can be quite complicated to work out the angles that you need to know from the information that you have. You often need to use the following geometric properties.

Opposite, corresponding and alternate angles

Where two lines intersect:

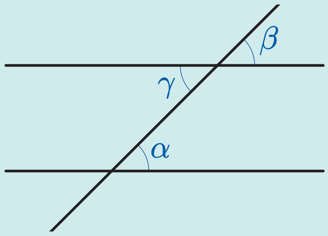
opposite angles are equal (for example, $\theta = \phi$).



Where a line intersects parallel lines:

corresponding angles are equal (for example, $\alpha = \beta$);

alternate angles are equal (for example, $\alpha = \gamma$).



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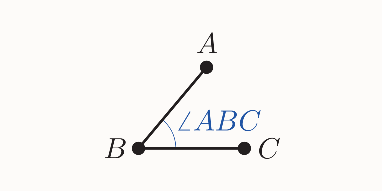


Figure 39 $\angle ABC$

The next example illustrates how to use some of these geometric properties. You should find the tutorial clip for this example particularly helpful.

The example uses the standard notation $\angle ABC$ (read as 'angle ABC') for the acute angle at the point B between the line segments AB and BC. This is illustrated in Figure 39.

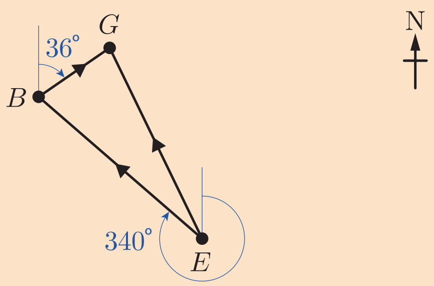
Example 13 *Adding two non-perpendicular vectors*

The displacement from Exeter to Belfast is 460 km with a bearing of 340°, and the displacement from Belfast to Glasgow is 173 km with a bearing of 36°. Use this information to find the magnitude (to the nearest kilometre) and direction (as a bearing, to the nearest degree) of the displacement from Exeter to Glasgow.

Solution

Denote Exeter by E, Belfast by B, and Glasgow by G.

Draw a diagram showing the displacement vectors $\overrightarrow{EB}$, $\overrightarrow{BG}$ and their resultant $\overrightarrow{EG}$. Mark the angles that you know. Mark or state any magnitudes that you know.



We know that $EB = 460$ km and $BG = 173$ km.

To enable you to calculate the magnitude and bearing of $\overrightarrow{EG}$, you need to find an angle in triangle BEG. Use geometric properties to find $\angle EBG$.

Since the bearing of $\overrightarrow{EB}$ is 340°, the acute angle at E between $\overrightarrow{EB}$ and north is $360° - 340° = 20°$, as shown in the diagram below.

Hence, since alternate angles are equal, the acute angle at B between $\overrightarrow{EB}$ and south is also 20°.

So $\angle EBG = 180° - 36° - 20° = 124°$.

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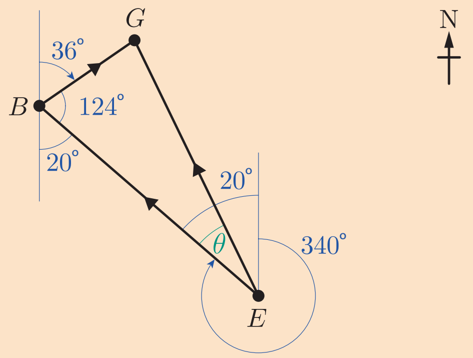

Now use the cosine rule to calculate EG.

Applying the cosine rule in triangle EBG gives

$$EG^2 = EB^2 + BG^2 - 2 \times EB \times BG \times \cos 124°,$$

so

$$\begin{aligned} EG &= \sqrt{460^2 + 173^2 - 2 \times 460 \times 173 \times \cos 124°} \\ &= 574.91\ldots = 575\,\text{km (to the nearest km)}. \end{aligned}$$

To find the bearing of $\overrightarrow{EG}$, first find $\angle BEG$.

Let $\angle BEG = \theta$, as marked in the diagram. Then, by the sine rule,

$$\frac{BG}{\sin\theta} = \frac{EG}{\sin 124°}$$

$$\begin{aligned} \sin\theta &= \frac{BG \sin 124°}{EG} \\ &= \frac{173 \sin 124°}{574.91\ldots}. \end{aligned}$$

Now

$$\sin^{-1}\left(\frac{173 \sin 124°}{574.91\ldots}\right) = 14.44\ldots°,$$

so

$$\theta = 14.44\ldots° \quad \text{or} \quad \theta = 180° - 14.44\ldots° = 165.55\ldots°.$$

If $\theta = 165.55\ldots°$, then the sum of θ and $\angle EBG$ (two of the angles in triangle EBG) is greater than 180°, which is impossible. So $\theta = 14.44\ldots°$.

Hence the bearing of $\overrightarrow{EG}$ is

$$340° + \theta = 340° + 14.44\ldots° = 354.44\ldots°.$$

State a conclusion.

The displacement of Glasgow from Exeter is 575 km (to the nearest km) on a bearing of 354° (to the nearest degree).

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As mentioned in Subsection 5.1, velocity is a vector quantity, since it is the speed with which an object is moving together with its direction of motion. So the methods that you have seen for adding displacements can also be applied to velocities.

It may at first seem strange to add velocities, but consider the following situation. Suppose that a boy is running across the deck of a ship. If the ship is motionless in a harbour, then the boy's velocity relative to the sea bed is the same as his velocity relative to the ship.

However, if the ship is moving, then the boy's velocity relative to the sea bed is a combination of his velocity relative to the ship and the ship's velocity relative to the sea bed. In fact, the boy's resultant velocity relative to the sea bed is the vector sum of the two individual velocities.

Activity 32 *Adding velocities*

A ship is steaming at a speed of $10.0\ \mathrm{m\,s^{-1}}$ on a bearing of $30°$ in still water. A boy runs across the deck of the ship from the port side to the starboard side, perpendicular to the direction of motion of the ship, with a speed of $4.0\,\mathrm{m\,s^{-1}}$ relative to the ship. (The port and starboard sides of a ship are the sides on the left and right, respectively, of a person on board facing the front.)

Find the resultant velocity of the boy, giving the speed in $\mathrm{m\,s^{-1}}$ to one decimal place and the bearing to the nearest degree.

When a ship sails in a current, or an aircraft flies through a wind, its actual velocity is the resultant of the velocity that it would have if the water or air were still, and the velocity of the current or wind. In particular, the direction in which the ship or aircraft is *pointing* – this is called its **heading**, when it is given as a bearing – may be different from the direction in which it is actually moving, which is called its **course**. This is because the current or wind may cause it to continuously drift to one side.

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Activity 33 *Finding the resultant velocity of a ship in a current*

A ship has a speed in still water of $5.7\,\mathrm{m\,s^{-1}}$ and is sailing on a heading of 230°. However, there is a current in the water of speed $2.5\,\mathrm{m\,s^{-1}}$ flowing on a bearing of 330°. Find the resultant velocity of the ship, giving the speed in $\mathrm{m\,s^{-1}}$ to one decimal place and the bearing to the nearest degree.

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